

# SUPERTROPICAL QUADRATIC FORMS II

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**ABSTRACT.** This article is a sequel of [4], where we introduced quadratic forms on a module  $V$  over a supertropical semiring  $R$  and analysed the set of bilinear companions of a quadratic form  $q : V \rightarrow R$  in case that the module  $V$  is free, with fairly complete results if  $R$  is a supersemifield. Given such a companion  $b$  we now classify the pairs of vectors in  $V$  in terms of  $(q, b)$ . This amounts to a kind of tropical trigonometry with a sharp distinction between the cases that a sort of Cauchy-Schwarz inequality holds or fails. We apply this to study the supertropicalizations (cf. [4]) of a quadratic form on a free module  $X$  over a field in the simplest cases of interest where  $\text{rk}(X) = 2$ .

In the last part of the paper we start exploiting the fact that the free module  $V$  as above has a unique base up to permutations and multiplication by units of  $R$ , and moreover  $V$  carries a so called minimal (partial) ordering. Under mild restriction on  $R$  we determine all  $q$ -minimal vectors in  $V$ , i.e., the vectors  $x \in V$  for which  $q(x') < q(x)$  whenever  $x' < x$ .

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## INTRODUCTION

Let  $R$  be a semiring, here always assumed to be commutative and with 1. A **quadratic form** on an  $R$ -module  $V$  is a function  $q : V \rightarrow R$  with

$$q(ax) = a^2q(x) \quad (0.1)$$

for any  $a \in R$ ,  $x \in V$ , such that there exists a symmetric bilinear form  $b : V \times V \rightarrow R$  (not necessarily uniquely determined by  $q$ ) with

$$q(x + y) = q(x) + q(y) + b(x, y) \quad (0.2)$$

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for any  $x, y \in V$ . Every such bilinear form  $b$  is called a **companion** of  $q$ , and the pair  $(q, b)$  is called a **quadratic pair** on  $V$ .

The present paper is devoted to a study of quadratic forms and pairs on  $R$ -modules with  $R$  a “supertropical” semiring, often more specifically a “supersemifield”. It is a sequel to the paper [4] by the same authors.

We recall ([4, Definition 0.3] and [1, §3]), that a semiring  $R$  is called **supertropical** if  $e := 1_R + 1_R$  is an idempotent (i.e.,  $2 \times 1 = 4 \times 1$ ), and the following axioms hold for all  $x, y \in R$ :

$$\text{If } ex \neq ey, \text{ then } x + y \in \{x, y\}, \quad (0.3)$$

$$\text{If } ex = ey, \text{ then } x + y = ey. \quad (0.4)$$

Then the ideal  $eR$  of  $R$  is a semiring with unit element  $e$ , which is **bipotent**, i.e., for any  $u, v \in eR$  the sum  $u + v$  is either  $u$  or  $v$ . It follows that  $eR$  carries a total ordering, compatible with addition and multiplication, which is given by

$$u \leq v \Leftrightarrow u + v = v. \quad (0.5)$$

The addition in a supertropical semiring is determined by the map  $x \mapsto ex$  and the total ordering on  $eR$  as follows: If  $x, y \in R$ , then

$$x + y = \begin{cases} y & \text{if } ex < ey, \\ x & \text{if } ex > ey, \\ ey & \text{if } ex = ey. \end{cases} \quad (0.6)$$

In particular (taking  $y = 0$  in (0.6) or in (0.4))

$$ex = 0 \Rightarrow x = 0. \quad (0.7)$$

For the convenience of the reader, we give more terminology. In a supertropical semiring  $R$ , the elements of the set  $\mathcal{T}(R) := R \setminus (eR)$  are called **tangible**, while those of the set  $\mathcal{G}(R) := (eR) \setminus \{0\}$  are called **ghost** elements. The zero of  $R$  is regarded both as tangible and ghost. The semiring  $R$  itself is called **tangible** if  $R$  is generated by  $\mathcal{T}(R)$  as a semiring. Clearly, this happens iff  $e\mathcal{T}(R) = \mathcal{G}(R)$ . If  $\mathcal{T}(R) \neq \emptyset$ , then the set

$$R' := \mathcal{T}(R) \cup e\mathcal{T}(R) \cup \{0\}$$

is the largest subsemiring of  $R$  which is tangible supertropical. {We have discarded the “superfluous” ghost elements.}

In the paper [4], the main thrust is the study of the set of all companions of a given quadratic form  $q$  on a free module  $V$  over a supertropical semiring  $R$ . After fixing a base  $(\varepsilon_i \mid i \in I)$  of  $V$ , this set can be described by use of a “companion matrix”  $(C_{i,j}(q))$ , cf. [4, §6]. For  $R$  a tangible semifield, complete results can be found in [4, §7]. Explicitly, these hard results are needed in the present paper only in the proof of the initial key Theorem 1.5, which for a first reading may be taken on faith.

The quadratic form  $q$  is called **rigid**, if  $q$  has only one companion. This happens iff  $q(\varepsilon_i) = 0$  for all vectors  $\varepsilon_i$  of the fixed base  $(\varepsilon_i \mid i \in I)$ , cf. [4, Theorem 3.5].  $q$  is called **quasilinear** if the bilinear form  $b = 0$  is a companion of  $q$ , i.e.,  $q(x + y) = q(x) + q(y)$  for all  $x, y \in V$ . These are the “diagonal” forms on  $V$ ,

$$q\left(\sum_i x_i \varepsilon_i\right) = \sum_i q(\varepsilon_i) x_i^2, \quad (0.8)$$

due to the fact that  $(\lambda + \mu)^2 = \lambda^2 + \mu^2$  for all  $\lambda, \mu \in R$ , cf. [4, Proposition 0.5].

Any quadratic form  $q$  on a free  $R$ -module can be written as a sum

$$q = q_{QL} + \rho, \quad (0.9)$$

where  $q_{QL}$  is a quasilinear (and uniquely determined by  $q$ ) and  $\rho$  is rigid (but not unique), cf. [4, §4]. We call  $q_{QL}$  the **quasilinear part** of  $q$  and  $\rho$  a **rigid complement** of  $q_{QL}$  in  $q$ .

The present paper is divided as follows. The first three sections are devoted to a study of pairs of non-zero vectors  $(x, y)$  in an  $R$ -module  $V$  equipped with a quadratic pair  $(q, b)$ , mostly for  $R$  a tangible semifield. Sometimes we only assume that  $eR$  is a (bipotent) semifield. We face an all important dichotomy. Either  $(x, y)$  is *excessive* (cf. Definition 1.6 below) or the restriction  $q|Rx + Ry$  of  $q$  is quasilinear. In the latter case, we also say that the pair  $(x, y)$  is **quasilinear** (with respect to  $q$ ).

An intriguing point here is that this dichotomy does not depend on the choice of the companion  $b$  of  $q$ , although  $b$  is used in the definition of excessiveness (cf. Corollary 1.7).

In Section 2, we delve into a kind of “*tropical trigonometry*”. If  $x$  and  $y$  are anisotropic, i.e.,  $q(x) \neq 0$ ,  $q(y) \neq 0$ , we define a CS-ratio<sup>1</sup>

$$\text{CS}(x, y) := \frac{eb(x, y)^2}{eq(x)q(y)} \in eR, \quad (0.10)$$

which makes sense since  $eR$  is a semifield. When the set  $eR$  is densely ordered, then  $(x, y)$  is excessive iff  $\text{CS}(x, y) > e$ . When  $eR$  is discrete, the pair  $(x, y)$  is excessive if  $\text{CS}(x, y) > c_0$ , with  $c_0$  the smallest element of  $eR$  bigger than  $e$ . But if  $\text{CS}(x, y) = c_0$ , the pair  $(x, y)$  is excessive if  $q(x)$  or  $q(y)$  is tangible, while  $(x, y)$  is quasilinear if both  $q(x)$  and  $q(y)$  are ghost (cf. Theorems 1.5 and 1.12). It seems to us that this still somewhat mysterious fact bears relevance for problems of an arithmetical nature in quadratic form theory, even over fields.

For any anisotropic vector  $w$ , the function  $x \mapsto \text{CS}(x, w)$  is subadditive, cf. Theorem 2.6. This fact has turned out to be of central importance in a (still incomplete) sequel [5] of the present paper.

In §3, we compile tables of the function  $(\lambda, \mu) \mapsto q(\lambda x + \mu y)$  on  $(R \setminus \{0\})^2$  for given  $x, y \in V \setminus \{0\}$ , and then study in detail the CS-ratios  $\text{CS}(x', y')$  of pairs of vectors  $(x', y')$  in  $Rx + Ry$ . This completes our account of tropical trigonometry in the present paper. First applications show up in the later sections, but a more adequate language of “rays”,<sup>2</sup> to use this trigonometry conveniently, has to wait for the paper [5] due to lack of space here.

Sections §4–§7 of the paper are based on the following two facts for  $R$ -modules, valid over any supertropical semiring  $R$ :

- 1) *The Unique Base Theorem*, cf. [4, Theorem 0.9]: Given a base  $(\varepsilon_i \mid i \in I)$  of a free  $R$ -module  $V$ , we obtain any other base of  $V$  by permuting the  $\varepsilon_i$  and multiplying them by units of  $R$ .
- 2) *Existence of minimal orderings*, cf. §5 below. Every  $R$ -module  $V$  carries a partial ordering, called the **minimal ordering** on  $V$ , which is defined as follows:

$$x \leq y \Leftrightarrow \exists z \in V : x + z = y.$$

In particular,  $R$  itself has a minimal ordering. The minimal ordering on  $V$  is compatible with addition and scalar multiplication. Basics about the minimal ordering on  $R$  and then on a free  $R$ -module are provided in §5.

<sup>1</sup>“CS” is an acronym of “Cauchy-Schwarz”.

<sup>2</sup>The ray of a vector  $x \in V \setminus \{0\}$  is the set of all  $y \in V$  with  $\lambda x = \mu y$  for some  $\lambda, \mu \in R \setminus \{0\}$ .

The Unique Base Theorem is the source of our motivation for introducing supertropicalizations of a quadratic form  $q : V \rightarrow R$  on a free module  $V$  over a ring  $R$  by a so-called *supervaluation*  $\varphi : R \rightarrow U$  with values in a supertropical semiring  $U$  in [4, §9]. Given a base  $\mathcal{L} = (\varepsilon_i \mid i \in I)$  of  $V$ , we obtained a quadratic form  $\tilde{q} : U^{(I)} \rightarrow U$  on the standard free  $U$ -module  $U^{(I)}$  by this process [loc. cit.], which in some sense measures  $\mathcal{L}$  in terms of  $q$  and  $\varphi$ . In §4 of the present paper, we study how  $\tilde{q}$  varies with a change of the base  $\mathcal{L}$  in the simplest cases of interest, where  $I = \{1, 2\}$ .

Given a quadratic form  $q : V \rightarrow R$  on a module  $V$  over a supertropical semiring  $R$ , we call a vector  $x \in V$   **$q$ -minimal**, if  $q(x') < q(x)$  for every vector  $x' < x$  (with respect to the minimal ordering of  $V$  and  $R$ ).<sup>3</sup>

In the last sections §6 and §7, we obtain a detailed description of all minimal vectors and certain relations between them in the case that  $V$  is free and  $R$  is tangible supertropical with  $\mathcal{G}(R)$  a cancellative monoid under multiplication (in particular, if  $R$  is a tangible supersemifield).

Every  $q$ -minimal vector  $x \in V$  is trapped in a smallest submodule  $V_J = \sum_{i \in J} Rv_i$  of  $V$  with  $|J| \leq 4$ , and thus it suffices to study  $q$ -minimal vectors in a given free module of rank at most 4. In §6 we easily find all  $q$ -minimal vectors for  $|J| \leq 2$  (vectors of “small support”). Then in §7 we prove that for  $|J| = 3$  or  $|J| = 4$  a  $q$ -minimal vector  $x$  is the maximum  $y \vee z$  of a pair of  $q$ -minimals  $y$  and  $z$  of small support which is uniquely determined by  $x$ , except in one case, where  $y$  and  $z$  can be freely chosen in a triplet  $y_1, y_2, y_3$  of  $q$ -minimals of small support, uniquely determined by  $x$ . Conversely, we find out which maxima  $y \vee z$  of  $q$ -minimals  $y, z$  with small support are again  $q$ -minimal.

The arguments in §6 and §7 may look massy due to the many case distinctions needed, but they give a good illustration of the, as we feel, beautiful combinatorics at hands in any supertropical quadratic space.

**Notation 0.1.** Let  $\mathbb{N} = \{1, 2, 3, \dots\}$ ,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . If  $R$  is a semiring, then  $R^*$  denotes the group of units of  $R$ .

If  $R$  is a supertropical semiring, then

- $\mathcal{T}(R) := R \setminus eR =$  set of tangible elements  $\neq 0$ .
- $\mathcal{G}(R) := eR \setminus \{0\} =$  set of ghost elements  $\neq 0$ .
- $\nu_R$  denotes the ghost map  $\rightarrow eR$ ,  $a \mapsto ea$ .

When there is no ambiguity, we write  $\mathcal{T}, \mathcal{G}, \nu$  instead of  $\mathcal{T}(R), \mathcal{G}(R), \nu_R$ .

For  $a \in R$  we also write  $ea = \nu(a) = a^\nu$ .  $a \leq_\nu b$  means that  $ea \leq eb$ ,  $a \cong_\nu b$  (“ $\nu$ -equivalent”) means that  $ea = eb$ , while  $a <_\nu b$  means that  $ea < eb$ .

## 1. PAIRS OF VECTORS IN A SUPERTROPICAL QUADRATIC SPACE

**Definition 1.1.**

- a) A **quadratic module** over a semiring  $R$  is a pair  $(V, q)$  consisting of an  $R$ -module  $V$  and a (functional) quadratic form  $q$  on  $V$ . Later we often will write a single letter  $V$  instead of  $(V, q)$ .
- b) A **supertropical quadratic space** is a quadratic module over a tangible supersemifield.

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<sup>3</sup>As usual,  $x' < x$  means  $x' \leq x$  and  $x' \neq x$ .

We intend to study pairs of vectors in a supertropical quadratic space. Preparing for this we slightly extend the notion of “partial” rigidity developed in [4, §3] (cf. [4, Definition 3.1]). This makes sense over any supertropical semiring  $R$ .

**Definition 1.2.** Let  $(V, q)$  be a quadratic module over a supertropical semiring  $R$ . We say that  $q$  is  **$\nu$ -rigid at a point**  $(x, y)$  of  $V \times V$  if

$$eb_1(x, y) = eb_2(x, y) \quad (1.1)$$

for any two companions  $b_1, b_2$  of  $q$ , and we say that  $q$  is  **$\nu$ -rigid on a set**  $T \subset V \times V$  or on a set  $S \subset V$ , if this happens for all  $(x, y)$  in  $T$  or in  $S \times S$ , respectively.

If the  $R$ -module  $V$  is free with base  $(\varepsilon_i \mid i \in I)$ , then  $\nu$ -rigidity of  $q$  at  $(\varepsilon_i, \varepsilon_j)$  means that all  $\beta \in C_{i,j}(q)$  have the same ghost value, i.e., the set  $e \cdot C_{i,j}(q)$  is a singleton. We have seen the phenomenon of  $\nu$ -rigidity (beyond rigidity) already in equation (6.5) of [4, Theorem 6.9].

Assume as before that  $R$  is a supertropical semiring  $R$ , and that  $(V, q)$  is a quadratic module over  $R$ . Given a pair of vectors  $(x, y) \in V \times V$ , we have a unique  $R$ -linear map

$$\chi := \chi_{x,y} : R\varepsilon_1 + R\varepsilon_2 \rightarrow V \quad (1.2)$$

from the free  $R$ -module  $R\varepsilon_1 + R\varepsilon_2$  with base  $\varepsilon_1, \varepsilon_2$  to  $V$  such that  $\chi(\varepsilon_1) = x$ ,  $\chi(\varepsilon_2) = y$ . This map  $\chi$  composes with  $q : V \rightarrow R$  to a quadratic form

$$\tilde{q} := q \circ \chi : R\varepsilon_1 + R\varepsilon_2 \rightarrow R. \quad (1.3)$$

**Proposition 1.3.**

- i) If  $b : V \times V \rightarrow R$  is a companion of  $q$ , then the symmetric bilinear form  $\tilde{b}$  on  $R\varepsilon_1 + R\varepsilon_2$  defined by

$$\tilde{b}(v_1, v_2) := b(\chi(v_1), \chi(v_2)) \quad (v_1, v_2 \in V) \quad (1.4)$$

is a companion of  $\tilde{q}$ .

- ii) If  $\tilde{q}$  is rigid at  $(\varepsilon_1, \varepsilon_2)$ , then  $q$  is rigid at  $(x, y)$ .  
 iii) If  $\tilde{q}$  is  $\nu$ -rigid at  $(\varepsilon_1, \varepsilon_2)$ , then  $q$  is  $\nu$ -rigid at  $(x, y)$ .

*Proof.* Claim i) follows directly from the definition of a companion in [4, §1] ([4, Definition 1.14]).

Claims ii) and iii) are immediate consequences of i). □

Concerning quasilinearity, we have a stronger statement.

**Proposition 1.4.** Given  $(x, y) \in V \times V$ , the following are equivalent.

- (i)  $q$  is quasilinear on  $Rx \times Ry$ .
- (ii)  $q$  is quasilinear on  $Rx + Ry$ .
- (iii)  $\tilde{q}$  is quasilinear on  $R\varepsilon_1 \times R\varepsilon_2$ .
- (iv)  $\tilde{q}$  is quasilinear.

*Proof.* Condition (iii) means that  $0 \in C_{1,2}(\tilde{q})$ . Since  $0 \in C_{i,i}(\tilde{q})$  holds for  $i = 1, 2$ , it is clear from [4, §5] that (iii)  $\Leftrightarrow$  (iv).

(ii) means that  $q$  is additive on  $Rx + Ry$ , while (iv) means that  $\tilde{q}$  is additive. Thus the equivalence (ii)  $\Leftrightarrow$  (iv) follows from the additivity and surjectivity of  $\chi$  as a map from  $R\varepsilon_1 + R\varepsilon_2$  to  $Rx + Ry$ .

(i) means that  $q(\lambda x + \mu y) = q(\lambda x) + q(\mu y)$ , and (iii) means that

$$\tilde{q}(\lambda \varepsilon_1 + \mu \varepsilon_2) = \tilde{q}(\lambda \varepsilon_1) + \tilde{q}(\mu \varepsilon_2)$$

for all  $\lambda, \mu \in R$  (cf. [4, Definition 2.3]). Thus clearly (i)  $\Leftrightarrow$  (iii).

We conclude that all four conditions (i) – (iv) are equivalent.  $\square$

We are ready for a key theorem of the paper, emanating from [4, §7].

**Theorem 1.5.** *Assume that  $R$  is a nontrivial tangible supersemifield and  $(q, b)$  is a quadratic pair on an  $R$ -module  $V$ . Let  $(x, y)$  be a pair of vectors in  $V$ . We adhere to [4, Terminology 7.7].*

a) *Assume that  $R$  is dense. Then  $q$  is quasilinear on  $Rx + Ry$  iff*

$$b(x, y)^2 \leq_\nu q(x)q(y). \quad (1.5)$$

*Otherwise  $q$  is rigid at  $(x, y)$ .*

b) *Assume that  $R$  is discrete with  $\pi$  a prime element of  $R$ .*

*Now  $q$  is quasilinear on  $Rx + Ry$  if either*

$$b(x, y)^2 <_\nu \pi^{-1}q(x)q(y) \quad (1.6)$$

*or both values  $q(x), q(y)$  are ghost and*

$$b(x, y)^2 \cong_\nu \pi^{-1}q(x)q(y). \quad (1.7)$$

*Otherwise  $q$  is  $\nu$ -rigid at  $(x, y)$ . If*

$$b(x, y)^2 >_\nu \pi^{-1}q(x)q(y) \quad (1.8)$$

*then  $q$  is rigid at  $(x, y)$ .*

*Proof.* By Propositions 1.3 and 1.4 above it suffices to prove these claims in the special case that  $V$  is free with base  $\varepsilon_1, \varepsilon_2$  and  $x = \varepsilon_1, y = \varepsilon_2$ . Now the results can be read off from [4, Proposition 7.9] and [4, Theorems 7.11 and 7.12].  $\square$

In order to obtain a better grasp on the contents of this theorem, we introduce more terminology. As before  $R$  is a nontrivial tangible supersemifield.

**Definition 1.6.** *Assume that  $(q, b)$  is a quadratic pair on an  $R$ -module  $V$ . We say that a pair of vectors  $(x, y) \in V \times V$  is **excessive** (w.r.t.  $(q, b)$ ), if the following holds:*

a) *If  $R$  is dense, then*

$$b(x, y)^2 >_\nu q(x)q(y).$$

b) *If  $R$  is discrete, then either*

$$b(x, y)^2 >_\nu \pi^{-1}q(x)q(y),$$

*or*

$$b(x, y)^2 \cong_\nu \pi^{-1}q(x)q(y)$$

*and  $q(x) \in \mathcal{T}$  or  $q(y) \in \mathcal{T}$ .*

Theorem 1.5, up to the rigidity statements there, can be reformulated as follows.

**Corollary 1.7.** *A pair  $(x, y) \in V \times V$  is excessive with respect to  $(q, b)$  iff  $q$  is **not** quasilinear on  $Rx + Ry$ .*

An intriguing point here is that the property “excessive” depends only on  $x, y, q$ . The choice of the companion  $b$  has no influence, but, of course, is relevant for deciding by computation whether  $(x, y)$  is excessive or not.

We state an easy consequence of Corollary 1.7.

**Proposition 1.8.** *Let  $(x, y)$  and  $(x', y')$  be pairs of vectors in a quadratic space  $(V, q)$  over a tangible supersemifield. Assume that  $(x', y')$  is excessive and  $Rx' + Ry' \subset Rx + Ry$ . Then  $(x, y)$  is excessive.*

*Proof.* Otherwise  $q$  would be quasilinear on  $Rx + Ry$ . But this implies that  $q$  is quasilinear on  $Rx' + Ry'$ , a contradiction.  $\square$

We now relax the assumption that  $R$  is a tangible supersemifield and demonstrate that several results obtained so far in the section remain valid in greater generality.

**Convention 1.9.** *We only assume that  $R$  is a supertropical semiring and  $eR$  is a semifield, i.e., every element of  $\mathcal{G} = eR \setminus \{0\}$  is invertible in  $eR$ ; hence  $\mathcal{G}$  is a totally ordered group. Moreover we assume that  $eR$  is “nontrivial”, i.e.,  $\mathcal{G} \neq \{e\}$ . We do not assume anything about  $\mathcal{T} := R \setminus eR$ . ( $\mathcal{T}$  may even be empty.) We call  $\mathcal{G}$  **discrete**, if  $\mathcal{G}$  contains a smallest element  $c > e$ , which we denote by  $c_0$ . (If  $R$  is a tangible supersemifield then  $c_0 = e\pi^{-1}$  in the setting [4, Terminology 7.7].) Otherwise we call  $\mathcal{G}$  **dense**.*

Assume in the following that  $(q, b)$  is a quadratic pair on the  $R$ -module  $V$ . For the sake of brevity we call a pair  $x, y$  of vectors in  $V \setminus \{0\}$  **quasilinear** if  $q$  is quasilinear on  $Rx \times Ry$ , equivalently if the restriction  $q|_{Rx \times Ry}$  of  $q$  is quasilinear.

**Definition 1.10.** *We say that a pair of vectors  $x, y$  in  $V \setminus \{0\}$  is **CS** (acronym for “Cauchy-Schwarz”), if*

$$b(x, y)^2 <_\nu q(x)q(y). \quad (1.9)$$

*We call  $(x, y)$  **weakly CS**, if*

$$b(x, y)^2 \leq_\nu q(x)q(y) \quad (1.10)$$

*(a condition already appearing in (1.5)), and we call  $(x, y)$  **almost CS**, if*

$$b(x, y)^2 \leq_\nu cq(x)q(y) \quad (1.11)$$

*for all  $c > e$  in  $\mathcal{G}$ .<sup>4</sup>*

**Remark 1.11.** *Assume that  $(x, y)$  is almost CS. If  $\mathcal{G}$  is dense, then  $(x, y)$  is weakly CS, whereas if  $\mathcal{G}$  is discrete, either  $(x, y)$  is weakly CS, or  $b(x, y)^2 \cong_\nu c_0 q(x)q(y)$ .*

We save a relevant part of Theorem 1.5 in the present more general situation.

**Theorem 1.12.** *If either  $(x, y)$  is weakly CS, or  $(x, y)$  is almost CS and both  $q(x)$  and  $q(y)$  are ghost, then  $(x, y)$  is quasilinear.*

*Proof.* If  $(x, y)$  satisfies the assumptions of the theorem then so does  $(\lambda x, \mu y)$  for all  $\lambda, \mu \in R \setminus \{0\}$ . Thus in view of Proposition 1.4 it suffices to prove that

$$q(x + y) = q(x) + q(y). \quad (*)$$

In general we have

$$q(x + y) = q(x) + q(y) + b(x, y). \quad (**)$$

<sup>4</sup>In [3, §5] the terms “CS” and “weakly CS” have been used in a similar way for pairs of vectors with respect to a (not necessarily symmetric) bilinear form.

If  $b(x, y)^2 <_\nu q(x)q(y)$ , then either  $b(x, y) <_\nu q(x)$  or  $b(x, y) <_\nu q(y)$ , and the summand  $b(x, y)$  in  $(**)$  can be omitted, giving  $(*)$ .

Assume now that  $b(x, y)^2 \cong_\nu q(x)q(y)$ . If  $q(x), q(y)$  are not  $\nu$ -equivalent, say  $q(x) <_\nu q(y)$ , then  $b(x, y)^2 <_\nu q(y)^2$ , hence  $b(x, y) <_\nu q(y)$ , and again the term  $b(x, y)$  can be omitted in  $(**)$ . If  $q(x) \cong_\nu q(y)$  then we have  $b(x, y)^2 \cong_\nu q(x)^2$ , hence  $b(x, y) \cong_\nu q(x) \cong_\nu q(y)$ , and the right hand side of  $(**)$  equals  $eq(x) = q(x) + q(y)$ . Thus  $(*)$  holds again.

There remains the case that  $\mathcal{G}$  is discrete and  $b(x, y)^2 \cong_\nu c_0 q(x)q(y)$ . Now  $q(x)q(y)$  is not a  $\nu$ -square. We may assume that  $q(x) <_\nu q(y)$ . Now  $c_0 q(x) \leq_\nu q(y)$ . Hence  $b(x, y)^2 \leq_\nu q(y)^2$ , and hence  $b(x, y) \leq_\nu q(y)$ . If  $b(x, y) <_\nu q(y)$  we obtain  $(*)$  from  $(**)$  as before. Otherwise  $b(x, y) \cong_\nu q(y)$ , and hence  $q(x + y) = eq(y) = q(x) + eq(y)$ . Thus, if  $q(y) \in eR$ , then  $q(x + y) = q(x) + q(y)$ .  $\square$

**Remark 1.13.** *The bad case is that  $R$  is discrete, with  $c_0 q(x) \cong_\nu q(y) \cong_\nu b(x, y)$ , perhaps after interchanging  $x$  and  $y$ , and  $q(y)$  is tangible. Then  $q(x) + q(y) = q(y)$ , while  $q(x + y) = q(y) + b(x, y) = eq(y)$ .*

**Remark 1.14.** *Let  $P$  be any of the properties in Definition 1.10 (CS, ...) or – if  $R$  is a tangible supersemifield – one of the conditions in Theorem 1.5. Assume that  $\lambda, \mu \in \mathcal{T}$ . Then it is obvious that a pair  $(x, y) \in V \times V$  has property  $P$  iff  $(\lambda x, \mu y)$  has property  $P$ . Except for the properties discussed in Theorem 1.5.b involving (1.7), this even remains true if  $\lambda, \mu \in R \setminus \{0\}$ .*

## 2. CS-RATIOS: DEFINITION AND SUBADDITIVITY

If  $R$  is any semiring and  $q : V \rightarrow R$  is a quadratic form on an  $R$ -module  $V$ , we call a vector  $x \in V \setminus \{0\}$  **isotropic** if  $q(x) = 0$  and **anisotropic** if  $q(x) \neq 0$ . The zero vector in  $V$  is regarded both as isotropic and anisotropic. If the semiring  $R$  is supertropical, it follows directly from the definition of a quadratic form (cf. [4, Eq. (0.1) and Eq. (0.2)]) that the set of anisotropic vectors

$$V_{\text{an}} := \{x \in V \mid q(x) \neq 0\} \cup \{0\}, \quad (2.1)$$

is an  $R$ -submodule of  $V$ , and moreover

$$V + V_{\text{an}} = V_{\text{an}}. \quad (2.2)$$

We now always assume in this section that  $R$  is supertropical, that  $eR$  is a **nontrivial bipotent semifield** (cf. Convention 1.9), and that  $(q, b)$  is a **fixed quadratic pair** on  $V$ . We develop the concept of “*CS-ratios*” for pairs of vectors in  $V_{\text{an}}$ . To a large extent this may be viewed as a kind of “trigonometry” in supertropical quadratic spaces.

We start with a definition where the quadratic pair is not yet needed.

**Definition 2.1.** *Given  $\lambda \in R$  and  $\mu \in R \setminus \{0\}$  the  $\nu$ -ratio  $\left[\frac{\lambda}{\mu}\right]_\nu$  is the fraction  $\frac{e\lambda}{e\mu}$  in the semifield  $eR = \mathcal{G} \cup \{0\}$ . Thus for any  $\gamma \in R$*

$$\left[\frac{\lambda}{\mu}\right]_\nu \cong_\nu \gamma \iff \gamma\mu \cong_\nu \lambda. \quad (2.3)$$

This slightly funny notation reflects the desire in supertropical algebra to work as much as possible with tangible elements. Indeed, if  $R$  happens to be a tangible supersemifield (the



most important case for us), we can write all  $\nu$ -ratios  $\neq 0$  as  $\left[\frac{\lambda}{\mu}\right]_\nu$  with  $\lambda, \mu \in \mathcal{T}$ . Then the  $\nu$ -ratio  $\left[\frac{\lambda}{\mu}\right]_\nu$  is characterized by

$$\forall \gamma \in \mathcal{T} : \quad \left[\frac{\lambda}{\mu}\right]_\nu \cong_\nu \gamma \iff \lambda \cong_\nu \gamma \mu. \quad (2.4)$$

**Definition 2.2.** Let  $x, y \in V_{\text{an}} \setminus \{0\}$ . We call

$$\text{CS}(x, y) := \left[\frac{b(x, y)^2}{q(x)q(y)}\right]_\nu$$

the **CS-ratio** of the pair of vectors  $(x, y)$  (with respect to  $(q, b)$ ).

**Remark 2.3.** In case of anisotropic vectors  $x, y$ , we can reformulate Definition 1.10 as follows: The pair  $(x, y)$  is CS iff  $\text{CS}(x, y) < e$ ; weakly CS iff  $\text{CS}(x, y) \leq e$ ; and almost CS iff  $\text{CS}(x, y) < c$  for any  $c > e$  in  $\mathcal{G}$ .

**Remark 2.4.** Clearly,  $\text{CS}(x, y) = \text{CS}(y, x)$ . Notice also that

$$\text{CS}(\lambda x, \mu y) = \text{CS}(x, y) \quad (2.5)$$

for any  $\lambda, \mu \in R \setminus \{0\}$ .

Given vectors  $x, y, w \in V_{\text{an}}$ , we look for constraints on the CS-ratio  $\text{CS}(x + y, w)$  in terms of  $\text{CS}(x, w)$  and  $\text{CS}(y, w)$ . We need a lemma from [6], (in fact a weak version of it), reproved here for the convenience of the reader.

**Lemma 2.5** (cf. [6, Lemma 3.16.ii]). Assume as before that  $eR$  is a semifield. Let  $a, b, c, d \in R$ .

i) If  $bc \cong_\nu ad$ , then

$$ac + bd = (a + b)(c + d). \quad (2.6)$$

ii) If  $a \cong_\nu b$ , **or**  $c \cong_\nu d$ , then still

$$ac + bd \cong_\nu (a + b)(c + d). \quad (2.7)$$

*Proof.* i): We assume without loss of generality that  $a \geq_\nu b$ .

1. Case:  $a \cong_\nu b \neq 0$ . Now  $c \cong_\nu d$ . Both sides of (2.6) equal  $ac$ .
2. Case:  $a >_\nu b$ . Now  $bc \cong_\nu ad$  implies that  $c >_\nu d$  or  $c = d = 0$ . If  $c = d = 0$ , both sides of (2.6) are zero. Otherwise  $ac >_\nu bd$ , and both sides of (2.6) equal  $ac$ .
3. Case:  $a = b = 0$ . Both sides of (2.6) are zero.

ii): This is evident. □

We are now ready for a theorem, which states subadditivity of the function  $x \mapsto \text{CS}(x, w)$  from  $V_{\text{an}} \setminus \{0\}$  to  $\mathcal{G}$  for a fixed  $w$ , together with refinements of this fact.

**Theorem 2.6.** Let  $x, y, w$  be anisotropic vectors in  $V$ .

a) Then

$$\text{CS}(x + y, w) \leq \text{CS}(x, w) + \text{CS}(y, w). \quad (2.8)$$

b) If  $q(x + y)$  is **not**  $\nu$ -equivalent to  $q(x) + q(y)$  and also  $\text{CS}(x, w) + \text{CS}(y, w) \neq 0$ , then

$$\text{CS}(x + y, w) < \text{CS}(x, w) + \text{CS}(y, w). \quad (2.9)$$

c) Assume that  $q(x + y) \cong_\nu q(x) + q(y)$ , and that either

$$\text{c1) } q(x) \text{CS}(y, w) = q(y) \text{CS}(x, w)$$

or

$$\text{c2) } \text{CS}(x, w) = \text{CS}(y, w)$$

*or*

$$\text{c3) } q(x) \cong_{\nu} q(y).$$

*Then*

$$\text{CS}(x + y, w) = \text{CS}(x, w) + \text{CS}(y, w). \quad (2.10)$$

*Proof.* Let  $c := \text{CS}(x, w)$ ,  $d := \text{CS}(y, w)$ . Thus

$$b(x, w)^2 \cong_{\nu} cq(x)q(w),$$

$$b(y, w)^2 \cong_{\nu} dq(y)q(w).$$

Adding these two relations and using that  $(\lambda + \mu)^2 = \lambda^2 + \mu^2$  for  $\lambda, \mu \in R$ , we obtain

$$b(x + y, w)^2 \cong_{\nu} [cq(x) + dq(y)]q(w). \quad (2.11)$$

Putting  $a := q(x)$ ,  $b := q(y)$ , we trivially have

$$ac + bd \leq_{\nu} (a + b)(c + d),$$

and further

$$a + b \leq_{\nu} q(x) + q(y) + b(x, y) = q(x + y).$$

We conclude that

$$b(x + y, w)^2 \leq_{\nu} (a + b)(c + d)q(w) \leq_{\nu} (c + d)q(x + y)q(w).$$

This tells us that  $\text{CS}(x + y, w) \leq c + d$ , which is claim a) of the theorem. Moreover, if  $a + b <_{\nu} q(x + y)$  and  $c + d \neq 0$ , then

$$b(x + y, w)^2 <_{\nu} (c + d)q(x + y)q(w),$$

which is claim b) of the theorem.

Henceforth we assume that  $q(x + y) \cong_{\nu} a + b$  and now have to prove equation (2.10). By (2.11) above the equation means that

$$ac + bd \cong_{\nu} (a + b)(c + d).$$

We know by Lemma 2.5 that this holds if  $ad \cong_{\nu} bc$ , and also if  $a \cong_{\nu} c$  or  $b \cong_{\nu} d$ . {We only need the statement (2.7) in the lemma, leaving the more interesting assertion (2.6) for later use.} This proves part c) of the theorem.  $\square$

### 3. A TABLE OF $q$ -VALUES, AND CS-RATIOS OF PAIRS OF VECTORS

Throughout this section  $V$  is a module over a tangible supersemifield  $R$ , and  $(q, b)$  is a quadratic pair on  $V$ . We fix a pair of vectors  $(x, y) \in V \times V$  and use the abbreviations

$$\alpha_1 := q(x), \quad \alpha_2 := q(y), \quad \alpha := b(x, y). \quad (3.1)$$

Our first goal is to compile a table of values of the function  $R \times R \rightarrow R$ ,  $(\lambda, \mu) \mapsto q(\lambda x + \mu y)$ , using the parameters  $\alpha_1, \alpha_2, \alpha$ . We then will use the table (Propositions 3.4 and 3.7) for various purposes here and in the sequels of this paper.

For establishing the table we may replace  $V$  by the free module  $R\varepsilon_1 + R\varepsilon_2$  with base  $\varepsilon_1, \varepsilon_2$ , the vector pair  $(x, y)$  by  $(\varepsilon_1, \varepsilon_2)$ , and the quadratic pair  $(q, b)$  by the quadratic pair  $(\tilde{q}, \tilde{b})$  on  $R\varepsilon_1 + R\varepsilon_2$ , obtained by composing  $(q, b)$  with the bilinear map

$$\chi : R\varepsilon_1 + R\varepsilon_2 \rightarrow V$$

with  $\chi(\varepsilon_1) = x$ ,  $\chi(\varepsilon_2) = y$ , as described in (1.2)–(1.4). Thus we may assume that  $V$  is free with base  $x, y$  and

$$q = \begin{bmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{bmatrix},$$

whenever we feel that this is convenient.

We do not assume this now, but we extend [4, Convention 7.10] for the parameters  $\alpha_1, \alpha_2, \alpha$  to the present situation in case that  $\alpha_1 \neq 0$  and  $\alpha_2 \neq 0$ . Thus we have an element  $\xi \in \mathcal{T}^{1/2}$  with  $\alpha_1 \xi \cong_\nu \alpha_2$ , and  $\xi \in \mathcal{T}$  if  $\alpha_1 \alpha_2$  is a  $\nu$ -square. In the case that  $R$  is discrete and  $\alpha_1 \alpha_2$  is not a  $\nu$ -square, we furthermore have elements  $\sigma, \tau$  in  $\mathcal{T}$ , such that  $e\tau < e\sigma$  and  $e\tau, e\sigma$  are the elements of  $\mathcal{G}$  nearest to  $e\xi$  in the totally ordered set  $\mathcal{G}^{1/2}$ , i.e.  $\tau <_\nu \xi <_\nu \sigma$  and  $\tau \cong_\nu \pi\sigma$ .

We enrich the setting of [4, Convention 7.10] as follows.

**Notation 3.1.** Assume that  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$ ,  $\alpha \neq 0$ . We choose  $\zeta, \eta \in \mathcal{T}$  with

$$\alpha \cong_\nu \zeta \alpha_1, \quad \alpha_2 \cong_\nu \eta \alpha, \quad (3.2)$$

and then have

$$\eta \zeta \cong_\nu \xi^2. \quad (3.3)$$

In the important special case that all three parameters  $\alpha_1, \alpha_2, \alpha$  are tangible, we take  $\zeta = \alpha \alpha_1^{-1}$ ,  $\eta = \alpha \alpha_2^{-1}$  and have

$$\alpha = \zeta \alpha_1, \quad \alpha_2 = \eta \alpha, \quad (3.2')$$

We then further arrange that

$$\eta \zeta = \xi^2. \quad (3.3')$$

**Remark 3.2.** Clearly  $\alpha^2 \cong_\nu \zeta \eta^{-1} \alpha_1 \alpha_2$ . Thus

$$\alpha_1 \alpha_2 <_\nu \alpha^2 \iff \eta <_\nu \zeta, \quad (3.4)$$

and then  $\eta <_\nu \xi <_\nu \zeta$ .

If in addition  $R$  is discrete and  $\xi \notin \mathcal{T}$ , then  $\alpha_1 \alpha_2 <_\nu \alpha^2$  implies that

$$\eta \leq_\nu \tau <_\nu \xi <_\nu \sigma \leq_\nu \zeta, \quad (3.5)$$

since  $e\tau, e\sigma$  are now the elements of  $\mathcal{G}$  nearest to  $e\xi \in \mathcal{G}^{1/2}$ . If even  $\alpha^2 >_\nu \pi^{-1} \alpha_1 \alpha_2$ , then

$$\eta <_\nu \tau <_\nu \xi <_\nu \sigma <_\nu \zeta. \quad (3.6)$$

**Convention 3.3.** Assuming again that  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$ ,  $\alpha \neq 0$ , we distinguish the following subcases of Cases I–III appearing in [4, Convention 7.10].

Case I:  $\alpha_1 \alpha_2$  is a  $\nu$ -square (i.e.  $\xi \in \mathcal{T}$ ).

IA:  $\alpha^2 >_\nu \alpha_1 \alpha_2$ , i.e.,  $\alpha >_\nu \xi \alpha_1$ .

IB:  $\alpha^2 \leq_\nu \alpha_1 \alpha_2$ .

Case II:  $R$  is dense, and  $\alpha_1 \alpha_2$  is not a  $\nu$ -square (hence  $\xi \notin \mathcal{T}$ ).

IIA:  $\alpha^2 >_\nu \alpha_1 \alpha_2$ , i.e.,  $\alpha >_\nu \xi \alpha_1$ .

IIB:  $\alpha^2 <_\nu \alpha_1 \alpha_2$ .

Case III:  $R$  is discrete, and  $\alpha_1 \alpha_2$  is not a  $\nu$ -square (hence  $\xi \notin \mathcal{T}$ ).

IIIA:  $\alpha^2 >_\nu \pi^{-1} \alpha_1 \alpha_2$ , i.e.,  $\eta <_\nu \tau <_\nu \sigma <_\nu \zeta$ .

IIIB:  $\alpha^2 \cong_\nu \pi^{-1} \alpha_1 \alpha_2$ , i.e.,  $\eta \cong_\nu \tau, \sigma \cong_\nu \zeta$ .

IIIC:  $\alpha^2 <_\nu \alpha_1 \alpha_2$ .

**Proposition 3.4.** Assume that  $\alpha_1, \alpha_2, \alpha$  are nonzero. Let  $\lambda, \mu \in R$ , not both zero.

(i) *In Cases IA, IIA, IIIA*

$$q(\lambda x + \mu y) = \begin{cases} \lambda^2 \alpha_1 & \lambda >_\nu \zeta \mu, \\ e\lambda^2 \alpha_1 = e\lambda \mu \alpha & \lambda \cong_\nu \zeta \mu, \\ \lambda \mu \alpha & \eta \mu <_\nu \lambda <_\nu \zeta \mu, \\ e\mu^2 \alpha_2 = e\lambda \mu \alpha & \lambda \cong_\nu \eta \mu, \\ \mu^2 \alpha_2 & \lambda <_\nu \eta \mu. \end{cases} \quad (3.7)$$

(ii) *In Case IIIB (now  $\zeta \cong_\nu \sigma$ ,  $\eta \cong_\nu \tau$ )*

$$q(\lambda x + \mu y) = \begin{cases} \lambda^2 \alpha_1 & \lambda >_\nu \sigma \mu, \\ e\lambda^2 \alpha_1 = e\lambda \mu \alpha & \lambda \cong_\nu \sigma \mu, \\ e\mu^2 \alpha_2 = e\lambda \mu \alpha & \lambda \cong_\nu \tau \mu, \\ \mu^2 \alpha_2 & \lambda <_\nu \tau \mu. \end{cases} \quad (3.8)$$

(iii) *In Case IB (hence  $\xi \in \mathcal{T}$ )*

$$q(\lambda x + \mu y) = \begin{cases} \lambda^2 \alpha_1 & \lambda >_\nu \xi \mu, \\ e\lambda^2 \alpha_1 = e\mu^2 \alpha_2 & \lambda \cong_\nu \xi \mu, \\ \mu^2 \alpha_2 & \lambda <_\nu \xi \mu. \end{cases} \quad (3.9)$$

(iv) *In Cases IIB, IIIC (hence  $\xi \notin \mathcal{T}$ )*

$$q(\lambda x + \mu y) = \begin{cases} \lambda^2 \alpha_1 & \lambda >_\nu \xi \mu, \\ \mu^2 \alpha_2 & \lambda <_\nu \xi \mu. \end{cases} \quad (3.10)$$

*Proof.* In Cases IB, IIB, IIIC the form  $q$  is quasilinear an  $Rx + Ry$ , as observed in Theorem 1.5, and hence

$$q(\lambda x + \mu y) = \lambda^2 \alpha_1 + \mu^2 \alpha_2,$$

and the claims in (3.9), (3.10) are immediate. In the other cases we have  $\alpha^2 >_\nu \alpha_1 \alpha_2$ ,  $\alpha \cong_\nu \zeta \alpha_1$ , and

$$q(\lambda x + \mu y) = \lambda^2 \alpha_1 + \lambda \mu \alpha + \mu^2 \alpha_2.$$

Now an easy inspection, which of the three terms on the right are  $\nu$ -dominant, gives us (3.7) and (3.8).  $\square$

It remains to handle the degenerate situation where at least one of the parameters  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha$  is zero.

**Convention 3.5.** *We distinguish the following cases, also for later use.*

Case IV:  $\alpha_1 \neq 0$ ,  $\alpha_2 = 0$ ,  $\alpha \neq 0$ .

Case V:  $\alpha_1 = \alpha_2 = 0$ ,  $\alpha \neq 0$ .

Case VI:  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$ ,  $\alpha = 0$ .

Case VII:  $\alpha_1 = \alpha_2 = \alpha = 0$ .

**Notations 3.6.** *In Case IV we choose  $\zeta \in \mathcal{T}$  with  $\alpha \cong_\nu \zeta \alpha_1$ . In the subcase that both  $\alpha_1, \alpha$  are tangible we take  $\zeta = \alpha \alpha_1^{-1}$  and then have  $\alpha = \zeta \alpha_1$ .*

Notice that the pair  $(x, y)$  is excessive in Cases IV, V, while  $q$  is quasilinear on  $Rx + Ry$  in the other two cases.

Now the following is obvious.

**Proposition 3.7.** *Let  $\lambda, \mu \in R$ , not both zero.*

(i) *In Case IV*

$$q(\lambda x + \mu y) = \begin{cases} \lambda^2 \alpha_1 & \text{if } \lambda >_\nu \zeta \mu, \\ e\lambda^2 \alpha_1 = e\lambda \mu \alpha & \text{if } \lambda \cong_\nu \zeta \mu, \\ \lambda \mu \alpha & \text{if } \lambda <_\nu \zeta \mu. \end{cases} \quad (3.11)$$

(ii) *In Case V*

$$q(\lambda x + \mu y) = \lambda \mu \alpha. \quad (3.12)$$

(iii) *In Case VI (3.9) holds if  $\xi \in \mathcal{T}$ , and (3.10) holds if  $\xi \notin \mathcal{T}$  (as in Cases IB resp. IIB, IIC).*

(iv) *In Case VII  $q(\lambda x + \mu y) = 0$ .*

**Remark 3.8.** *The tables in Proposition 3.4 and 3.7 reveal that (for fixed  $x, y$ ) the  $\nu$ -value of  $q(\lambda x + \mu y)$  only depends on the  $\nu$ -values of  $\lambda$  and  $\mu$ . This is conceptually evident from the equation*

$$eq(\lambda x + \mu y) = q((e\lambda)x + (e\mu)y).$$

We now use these tables to compute the CS-ratios of pairs of vectors in  $Rx + Ry$  in the case that the pair  $(x, y)$  is free and excessive.

**Convention 3.9.** *Assume that the submodule  $Rx + Ry$  of  $V$  is free with base  $x, y$ , and that the pair  $(x, y)$  is excessive. Let  $x', y' \in Rx + Ry$  be given with  $x' \neq 0$ ,  $y' \neq 0$ ,  $x' \neq y'$ . We write*

$$x' = \lambda_1 x + \mu_1 y, \quad y' = \lambda_2 x + \mu_2 y, \quad (3.13)$$

*with  $\lambda_i, \mu_i \in R$ . We exclude the (trivial) case that  $\mathcal{G}x' = \mathcal{G}y'$  and assume without loss of generality that*

$$\lambda_1 \mu_2 >_\nu \lambda_2 \mu_1, \quad (3.14)$$

*which for  $\mu_1 \neq 0$  means that  $\left[\frac{\lambda_1}{\mu_1}\right]_\nu > \left[\frac{\lambda_2}{\mu_2}\right]_\nu$ . (Recall Definition 2.1.) Since the pair  $(x, y)$  is free and excessive, the symmetric bilinear form  $b$  on  $Rx + Ry$  with*

$$b(x, x) = b(y, y) = 0, \quad b(x, y) = \alpha, \quad (3.15)$$

*is a companion of  $q|_{Rx + Ry}$ .*

**Problem 3.10.**

- Compute the CS-ratio  $\text{CS}(x', y')$  with respect to  $(q, b)$  in terms of  $\alpha_1, \alpha_2, \alpha$  and the  $\lambda_i, \mu_j$ , if both  $x', y'$  are anisotropic.*
- Decide which pairs  $(x', y')$  are again excessive.*

We know by Corollary 1.7 that in case that  $(x', y')$  is *not* excessive, the quadratic form  $q$  is quasilinear on  $Rx' + Ry'$ . We then say in brief that the pair  $(x', y')$  is quasilinear.

In the following we write  $\cong$  instead of  $\cong_\nu$  and  $\left[\frac{\lambda}{\mu}\right]$  instead of  $\left[\frac{\lambda}{\mu}\right]_\nu$ , for short.

As a consequence of (3.14) and (3.15) we have

$$b(x', y') = \lambda_1 \mu_2 \alpha. \quad (3.16)$$

Since  $(x', y')$  is excessive, we are in one of the Cases IA, IIA, IIIA, IIIB, IV, V, and in Case IIIB if at least one of the elements  $\alpha_1, \alpha_2$  is tangible (cf. Definition 1.6).

We postpone the degenerate Cases IV and V, and thus assume now that  $\alpha_1 \neq 0, \alpha_2 \neq 0$ , and  $\alpha^2 <_\nu \alpha_1 \alpha_2$ . We constantly use the table in Proposition 3.4, based on Notation 3.1, and rely heavily on Theorem 1.5.

Before entering systematic computations, we warm up with some observations. We have

$$\text{CS}(x, y) = \left[ \frac{\alpha^2}{\alpha_1 \alpha_2} \right] = \left[ \frac{\zeta^2 \alpha_1^2}{\zeta \eta \alpha_1^2} \right] = \left[ \frac{\zeta}{\eta} \right]. \quad (3.17)$$

The vectors

$$z := \zeta x + y, \quad w := \eta x + y \quad (3.18)$$

will play a prominent role. We have

$$q(z) = e\zeta^2 \alpha_1, \quad q(w) = e\alpha_2, \quad b(z, w) = \zeta \alpha = \zeta^2 \alpha_1;$$

hence

$$\text{CS}(z, w) = \left[ \frac{\zeta^4 \alpha_1^2}{\zeta^2 \alpha_1 \alpha_2} \right] = \left[ \frac{\zeta^2 \alpha_1^2}{\zeta \eta \alpha_1^2} \right] = \left[ \frac{\zeta}{\eta} \right] >_\nu 1. \quad (3.19)$$

We conclude that the pair  $(z, w)$  is excessive, except in Case IIIB. Then  $(z, w)$  is quasilinear, since both  $q(z), q(w)$  are ghost. On the other hand

$$\text{CS}(x, z) = \left[ \frac{\alpha^2}{\alpha_1 \cdot \zeta^2 \alpha_1} \right] \cong_\nu 1, \quad (3.20)$$

$$\text{CS}(w, y) = \left[ \frac{\eta^2 \alpha^2}{\alpha_1 \cdot \alpha_2} \right] \cong_\nu 1. \quad (3.21)$$

Thus both pairs  $(x, z), (w, y)$  are quasilinear.

The CS-values (3.17) and (3.19)–(3.21) make it plausible that

$$\text{CS}(x', y') \leq \left[ \frac{\zeta}{\eta} \right] = \text{CS}(x, y) \quad (3.22)$$

for all pairs  $(x', y')$  in  $(Rx + Ry) \setminus \{0\}$ . This is indeed true, as we will verify below.

We are ready to compute  $\text{CS}(x', y')$  in all cases.

a) Assume that  $\lambda_2 \geq_\nu \zeta \mu_2$ . Now  $q(x') \cong \lambda_1^2 \alpha_1, q(y') \cong \lambda_2^2 \alpha$ , and hence

$$\text{CS}(x', y') = \left[ \frac{\lambda_1^2 \mu_2^2 \alpha^2}{\lambda_1^2 \alpha_1 \lambda_2^2 \alpha} \right] = \left[ \frac{\mu_2^2 \zeta^2}{\lambda_2^2} \right] \leq_\nu 1. \quad (3.23)$$

N.B. This is smaller than  $1^\nu$  if  $\lambda_2 >_\nu \zeta \mu_2$ .

b) Assume that  $\lambda_1 \leq_\nu \eta \mu_1$ . We obtain

$$\text{CS}(x', y') = \left[ \frac{\lambda_1^2 \mu_2^2 \alpha^2}{\mu_1^2 \alpha_1 \mu_2^2 \alpha_2} \right] = \left[ \frac{\lambda_1^2 \alpha^2}{\mu_1^2 \alpha_2} \right] = \left[ \frac{\lambda_1^2}{\mu_1^2 \eta^2} \right] \leq_\nu 1. \quad (3.24)$$

This can also be deduced from a) by interchanging  $x, y$  and  $x', y'$ .

c) Assume that  $\lambda_1 \leq_\nu \zeta \mu_1, \lambda_2 \geq_\nu \eta \mu_2$ .<sup>5</sup> Now  $q(x') \cong \lambda_1 \mu_1 \alpha, q(y') \cong \lambda_2 \mu_2 \alpha$ , and hence

$$\text{CS}(x', y') = \left[ \frac{\lambda_1^2 \mu_2^2 \alpha^2}{\lambda_1 \lambda_2 \mu_1 \mu_2 \alpha^2} \right] = \left[ \frac{\lambda_1 \mu_2}{\mu_1 \lambda_2} \right] >_\nu 1. \quad (3.25)$$

---

<sup>5</sup>Perhaps the most interesting case!

Thus  $(x', y')$  is excessive except in Case IIIB. Then there exist no  $\nu$ -values in  $R$  strictly between  $\zeta$  and  $\eta$ . Hence  $\text{CS}(x', y') = \text{CS}(z, w)$ , and we know from the above that  $(x', y')$  is quasilinear.

d)  $\lambda_1 \geq_\nu \zeta\mu_1$ ,  $\zeta\mu_2 \geq_\nu \lambda_2 \geq_\nu \eta\mu_2$ . We obtain in the same way

$$\text{CS}(x', y') = \left[ \frac{\zeta\mu_2}{\lambda_2} \right] = \text{CS}(z, y'). \quad (3.26)$$

e)  $\zeta\mu_1 \geq_\nu \lambda_1 \geq_\nu \eta\mu_1$ ,  $\lambda_2 \leq_\nu \eta\mu_2$ . We obtain

$$\text{CS}(x', y') = \left[ \frac{\lambda_1}{\mu_1\eta} \right] = \text{CS}(x', w). \quad (3.27)$$

Again we can also deduce e) from d) by interchanging  $x, y$  and  $x', y'$ .

f) The degenerate Case IV, where  $\alpha_1 \neq 0$ ,  $\alpha_2 = 0$ , and only one parameter  $\zeta$  is present. We obtain

$$\text{CS}(x', y') = \begin{cases} \left[ \frac{\mu_2^2\zeta^2}{\lambda_2^2} \right] \leq_\nu 1 & \text{if } \lambda_2 \geq_\nu \zeta\mu_2; \\ \left[ \frac{\mu_2\zeta}{\lambda_2} \right] >_\nu 1 & \text{if } \lambda_1 \geq_\nu \zeta\mu_1 \text{ and } \lambda_2 <_\nu \zeta\mu_2; \\ \left[ \frac{\lambda_1\mu_2}{\lambda_2\mu_1} \right] >_\nu 1 & \text{if } \lambda_1 \leq_\nu \zeta\mu_1. \end{cases} \quad (3.28)$$

g)  $\alpha_1 = \alpha_2 = 0$ ,  $\alpha \neq 0$  (Case V). Now

$$\text{CS}(x', y') = \left[ \frac{\lambda_1^2\mu_2^2\alpha^2}{\lambda_1\mu_1\alpha \cdot \lambda_2\mu_2\alpha} \right] = \left[ \frac{\lambda_1\mu_2}{\mu_1\lambda_2} \right] >_\nu 1. \quad (3.29)$$

Thus every pair  $(x', y')$  is excessive.

Equations (3.28) and (3.29) may be viewed as resulting from (3.23)–(3.27) by putting  $\eta = 0$  and  $\eta = 0, \zeta = \infty$ , respectively.

The solution just obtained for Problem 3.10.b remains valid if the pair  $(x, y)$  is not necessarily free, since we can choose a linear map  $\chi : R\varepsilon_1 + R\varepsilon_2 \rightarrow V$  with  $\chi(\varepsilon_1) = z$ ,  $\chi(\varepsilon_2) = y$  as in (1.2), define  $\tilde{q}, \tilde{b}$  on the free module  $R\varepsilon_1 + R\varepsilon_2$  as in (1.3), where now  $b$  is any companion of  $q$  and then apply Proposition 1.4. {Notice that the value  $\alpha = b(x, y)$  does not depend on the choice of  $b$ , since  $(x, y)$  is excessive.} We thus arrive at the following theorem.

**Theorem 3.11.** *Continuing with Notations 3.1 and 3.6, we assume that the pair  $(x, y)$  is excessive, and without loss of generality, that either  $\alpha_1 \neq 0$  or  $\alpha_1 = \alpha_2 = 0$ . Let  $x' = \lambda_1x + \mu_1y$ ,  $y' = \lambda_2x + \mu_2y$  with  $\lambda_i, \mu_i \in R$  and  $\lambda_1\mu_2 >_\nu \lambda_2\mu_1$ . Then  $q$  is quasilinear on  $Rx' + Ry'$  precisely in the following three cases.*

- 1)  $\alpha_1 \neq 0, \lambda_2 \geq_\nu \zeta\mu_2$ ;
- 2)  $\alpha_1 \neq 0, \alpha_2 \neq 0, \lambda_1 \leq_\nu \eta\mu_1$ ;
- 3)  $\alpha_1 \neq 0, \alpha_2 \neq 0$ ,  $R$  discrete,  $\lambda_1 \cong_\nu \zeta\mu_1, \lambda_2 \cong_\nu \eta\mu_2$ .

Otherwise  $(x', y')$  is excessive.

#### 4. SUPERTROPICALIZATION: TWO EXAMPLES

We illustrate the dependence of the stropicalization  $q^\varphi$  of a quadratic form  $q : V \rightarrow R$  on the choice of a base of the free module  $V$  by two examples, which may be regarded as the simplest cases of interest.

We assume that  $R$  is a field and  $\varphi : R \rightarrow U$  is a supervaluation [1, §4]. Let  $v : R \rightarrow M := eU$  denote the valuation covered by  $\varphi$ . Leaving aside a less interesting case, we assume that  $v \neq e\varphi$ , i.e.,  $e \neq 1_U$ . Then  $\varphi$  is “tangible”, i.e., all values  $\varphi(a)$ ,  $a \in R$ , are tangible [1, Proposition 8.13]. Making  $U$  smaller we may assume, without loss of generality, that  $\varphi(R^*) = \mathcal{T}$ ,  $v(R^*) = \mathcal{G}$ , with  $\mathcal{T} := \mathcal{T}(U)$ ,  $\mathcal{G} = \mathcal{G}(U)$ . Now  $U$  is a tangible supersemifield.

We further assume that the supervaluation  $\varphi$  is “tangibly additive” [1, Definition 9.6].<sup>6</sup> Since  $R$  is a ring, even a field, this implies that  $\varphi$  is “very strong” [1, §10], i.e., for all  $a, b \in R$ ,

$$v(a) < v(b) \Rightarrow \varphi(a + b) = \varphi(b). \quad (4.1)$$

We briefly recall the process of stropicalization when  $\dim V = 2$ . Let

$$q = (q, v_1, v_2) = \begin{bmatrix} \alpha_1 & \alpha \\ & \alpha_2 \end{bmatrix}$$

denote the presentation of the given (functional) quadratic form  $q : V \rightarrow R$  after choice of a base  $v_1, v_2$  of the vector space  $V$ . Then

$$q^\varphi := (q, v_1, v_2)^\varphi = \begin{bmatrix} \varphi(\alpha_1) & \varphi(\alpha) \\ & \varphi(\alpha_2) \end{bmatrix} \quad (4.2)$$

is the stropicalization of  $q$  with respect to  $(v_1, v_2)$ , and

$$b^\varphi = (b, v_1, v_2)^\varphi = \begin{pmatrix} \varphi(2)\varphi(\alpha_1) & \varphi(\alpha) \\ \varphi(\alpha) & \varphi(2)\varphi(\alpha_2) \end{pmatrix} \quad (4.3)$$

is the stropicalization of the unique companion  $b : V \times V \rightarrow R$  of  $q$  with respect to  $(v_1, v_2)$ , cf. [4, Eq. (9.14) and Eq. (9.15)]. Here the presentations (4.2), (4.3) refer to the standard base  $\varepsilon_1, \varepsilon_2$  of  $U^2$ .

**Example A.**

$$(q, v_1, v_2) = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}. \text{ We take a new base of } V$$

$$v'_1 = a_{11}v_1 + a_{12}v_2, \quad v'_2 = a_{21}v_1 + a_{22}v_2,$$

and then have

$$(q, v'_1, v'_2) = \begin{bmatrix} a_{11}a_{12} & a_{11}a_{22} + a_{12}a_{21} \\ & a_{21}a_{22} \end{bmatrix}.$$

We abbreviate

$$\tilde{q} := (q, v'_1, v'_2)^\varphi, \quad \tilde{b} := (b, v'_1, v'_2)^\varphi.$$

Thus

$$\tilde{q} = \begin{bmatrix} \varphi(a_{11}a_{12}) & \varphi(a_{11}a_{22} + a_{12}a_{21}) \\ & \varphi(a_{21}a_{22}) \end{bmatrix}. \quad (4.4)$$

**Case I:**  $\varphi(a_{11}a_{22}) >_\nu \varphi(a_{12}a_{21})$ .

Using (4.1) we obtain from (4.4)

$$\tilde{q} = \begin{bmatrix} \varphi(a_{11}a_{12}) & \varphi(a_{11}a_{22}) \\ & \varphi(a_{21}a_{22}) \end{bmatrix}.$$

---

<sup>6</sup>In general, the tangibly additive supervaluations seem to be the most suitable ones for applications, cf. [1, §9-§11].



This implies (cf. (4.2), (4.3))

$$\tilde{b} = \begin{pmatrix} \varphi(2a_{11}a_{12}) & \varphi(a_{11}a_{22}) \\ \varphi(a_{11}a_{22}) & \varphi(2a_{21}a_{22}) \end{pmatrix}.$$

Thus we have

$$\tilde{q} = c \begin{bmatrix} b_1 & 1 \\ & b_2 \end{bmatrix}, \quad \tilde{b} = c \begin{pmatrix} \varphi(2)b_1 & 1 \\ 1 & \varphi(2)b_2 \end{pmatrix},$$

with

$$\begin{aligned} c &:= \varphi(a_{11}a_{22}) \in \mathcal{T}, \\ b_1 &:= \frac{\varphi(a_{12})}{\varphi(a_{22})} \in \mathcal{T} \cup \{0\}, \\ b_2 &:= \frac{\varphi(a_{21})}{\varphi(a_{11})} \in \mathcal{T} \cup \{0\}, \end{aligned}$$

and  $0 < \varphi(2) \leq e$  if  $\text{char } R \neq 2$ , while  $\varphi(2) = 0$  if  $\text{char } R = 2$ . The value  $\varphi(2)$  will not matter in what follows.

Notice that  $b_1b_2 <_\nu 1$ . Thus the pair  $(\varepsilon_1, \varepsilon_2)$  is excessive (cf. Definition 1.6). If  $b_1b_2 \neq 0$ , we have the CS-ratio

$$\text{CS}(\varepsilon_1, \varepsilon_2) \cong_\nu \frac{1}{b_1b_2}$$

with respect to  $(\tilde{q}, \tilde{b})$ .

**Case II:**  $\varphi(a_{11}a_{22}) <_\nu \varphi(a_{12}a_{21})$ .

Now

$$\tilde{q} = \begin{bmatrix} \varphi(a_{11}a_{12}) & \varphi(a_{12}a_{21}) \\ & \varphi(a_{21}a_{22}) \end{bmatrix},$$

and we obtain

$$\tilde{q} = c \begin{bmatrix} b_1 & 1 \\ & b_2 \end{bmatrix}, \quad \tilde{b} = c \begin{pmatrix} \varphi(2)b_1 & 1 \\ 1 & \varphi(2)b_2 \end{pmatrix}$$

with  $c, b_1, b_2$  as above (Case I). Again  $b_1b_2 <_\nu 1$ , whence  $(\varepsilon_1, \varepsilon_2)$  is excessive and, if  $b_1b_2 \neq 0$ ,

$$\text{CS}(\varepsilon_1, \varepsilon_2) \cong_\nu \frac{1}{b_1b_2}.$$

**Case III:**  $\varphi(a_{11}a_{22}) \cong_\nu \varphi(a_{12}a_{21}) \neq 0$ .

Using the general rule

$$v(x + y) \leq v(x) + v(y) \quad [= \max\{v(x), v(y)\}]$$

(cf. [1, Definition 2.1]) for  $m$ -valuations, we obtain from (4.4)

$$\tilde{q} = \begin{bmatrix} \varphi(a_{11}a_{12}) & \gamma \\ & \varphi(a_{21}a_{22}) \end{bmatrix}$$

with  $\gamma \leq_\nu \varphi(a_{11}a_{12})$ , and then

$$\tilde{q} = c \begin{bmatrix} b_1 & \delta \\ & b_2 \end{bmatrix}, \quad \tilde{b} = c \begin{pmatrix} \varphi(2)b_1 & \delta \\ \delta & \varphi(2)b_2 \end{pmatrix}$$

with

$$c = \varphi(a_{11}a_{22}) \in \mathcal{T}, \quad \delta \leq_\nu 1, \\ b_1 = \frac{\varphi(a_{12})}{\varphi(a_{22})} \in \mathcal{T}, \quad b_2 = \frac{\varphi(a_{21})}{\varphi(a_{11})} \in \mathcal{T}.$$

Now  $\delta^2 \leq_\nu 1 = b_1b_2$ . Thus the pair  $(\varepsilon_1, \varepsilon_2)$  is quasilinear,<sup>7</sup> whence

$$\tilde{q} = [b_1, b_2] := \begin{bmatrix} b_1 & 0 \\ & b_2 \end{bmatrix}.$$

More precisely,  $(\varepsilon_1, \varepsilon_2)$  is weakly CS with respect to  $(\tilde{q}, \tilde{b})$  (cf. Definition 1.10).

These three cases exhaust all possibilities, since we cannot have  $a_{11}a_{22} = a_{12}a_{21} = 0$ , because  $a_{11}a_{22} - a_{12}a_{21} \neq 0$ .

**Example B.**

$$(q, v_1, v_2) = [\alpha, \beta] := \begin{bmatrix} \alpha & 0 \\ & \beta \end{bmatrix} \text{ with } \alpha \neq 0, \beta \neq 0.$$

We choose a new base

$$v'_1 = a_{11}v_1 + a_{12}v_2, \quad v'_2 = a_{21}v_1 + a_{22}v_2$$

of  $V$ . Then

$$(q, v'_1, v'_2) = \begin{bmatrix} a_{11}^2\alpha + a_{12}^2\beta & a_{11}a_{21}\alpha + a_{12}a_{22}\beta \\ & a_{21}^2\alpha + a_{22}^2\beta \end{bmatrix}. \quad (4.5)$$

We use again the abbreviations

$$\tilde{q} := (q, v'_1, v'_2)^\varphi, \quad \tilde{b} := (b, v'_1, v'_2)^\varphi.$$

**Case I:**  $v(a_{11}^2\alpha) > v(a_{12}^2\beta), \quad v(a_{21}^2\alpha) > v(a_{22}^2\beta)$ .

It follows that  $a_{11} \neq 0, a_{21} \neq 0$ , and

$$v(a_{11}a_{21}\alpha) > v(a_{12}a_{22}\beta).$$

Thus

$$\tilde{q} = \begin{bmatrix} \varphi(a_{11}^2\alpha) & \varphi(a_{11}a_{21}\alpha) \\ & \varphi(a_{21}^2\alpha) \end{bmatrix}.$$

We conclude that  $\text{CS}(\varepsilon_1, \varepsilon_2) = e$ , and hence the pair  $(\varepsilon_1, \varepsilon_2)$  is quasilinear (more precisely, weakly CS), whence

$$\tilde{q} = [\varphi(a_{11}^2\alpha), \varphi(a_{21}^2\alpha)] = [\varphi(\alpha), \varphi(\alpha)]. \quad (4.6)$$

**Case II:**  $v(a_{11}^2\alpha) = v(a_{12}^2\beta), \quad v(a_{21}^2\alpha) > v(a_{22}^2\beta)$ .

We have  $a_{11} \neq 0, a_{12} \neq 0$ , since otherwise  $a_{11} = a_{12} = 0$ , which contradicts  $a_{11}a_{22} - a_{12}a_{21} \neq 0$ . It follows that

$$v(a_{11}a_{21}\alpha) > v(a_{12}a_{22}\beta),$$

and we obtain from (4.5)

$$\tilde{q} = \begin{bmatrix} \varphi(a_{11}^2\alpha + a_{12}^2\beta) & \varphi(a_{11}a_{12}\alpha) \\ & \varphi(a_{21}^2\alpha) \end{bmatrix}.$$

---

<sup>7</sup>By this we mean that  $\tilde{q}$  is quasilinear on  $U\varepsilon_1 \times U\varepsilon_2$ , hence on  $U\varepsilon_1 + U\varepsilon_2$ , cf. §1.

**Case III:**  $v(a_{11}^2\alpha) = v(a_{12}^2\beta)$ ,  $v(a_{21}^2\alpha) = v(a_{22}^2\beta)$ .

From  $a_{11}a_{22} - a_{21}a_{12} \neq 0$  we conclude that all entries  $a_{ij} \neq 0$ . We cannot say more in this generality.

Note: In Cases II and III the nature of the pair  $(\varepsilon_1, \varepsilon_2)$  (excessive, quasilinear, weakly CS, ...) remains undetermined by the values  $v(a_{ij})$ . This may change if we have specified information about the value  $v$  and  $\alpha$  and  $\beta$ . For example, if  $v$  is compatible with a total ordering  $\leq$  of the field  $R$ , cf. [8], and  $\alpha > 0$ ,  $\beta > 0$ , then

$$v(a_{11}^2\alpha + a_{12}^2\beta) = \max\{v(a_{11}^2\alpha), v(a_{12}^2\beta)\}$$

and we can say more. It may well happen that  $\tilde{q}$  is not quasilinear. (We do not go into details.)

Notice that Cases II and III can only occur if  $v(\alpha)$ ,  $v(\beta)$  are square equivalent (cf. [4, Definition 7.1]).

**Case IV:**  $v(a_{11}^2\alpha) > v(a_{12}^2\beta)$ ,  $v(a_{21}^2\alpha) < v(a_{22}^2\beta)$ .

Now we read off from (4.5) that

$$\tilde{q} = \begin{bmatrix} \varphi(a_{11}^2\alpha) & \varphi(a_{11}a_{21}\alpha + a_{12}a_{22}\beta) \\ & \varphi(a_{22}^2\beta) \end{bmatrix}.$$

We have  $a_{11} \neq 0$ ,  $a_{22} \neq 0$ . Thus the CS-ratio  $\text{CS}(\varepsilon_1, \varepsilon_2)$  exists and

$$\begin{aligned} \text{CS}(\varepsilon_1, \varepsilon_2) &= \frac{v(a_{11}a_{21}\alpha + a_{12}a_{22}\beta)^2}{v(a_{11}^2\alpha)v(a_{22}^2\beta)} \\ &\leq \frac{v(a_{11}^2\alpha^2)}{v(a_{11}^2\alpha)v(a_{22}^2\beta)} + \frac{v(a_{12}^2\beta^2)}{v(a_{11}^2\alpha)v(a_{22}^2\beta)} \\ &= \frac{v(a_{21}^2\alpha)}{v(a_{22}^2\beta)} + \frac{v(a_{12}^2\beta)}{v(a_{11}^2\alpha)} < e. \end{aligned}$$

Thus the pair  $(\varepsilon_1, \varepsilon_2)$  is CS (cf. Definition 1.10) and

$$\tilde{q} = [\varphi(a_{11}^2\alpha), \varphi(a_{22}^2\beta)] = [\varphi(\alpha), \varphi(\beta)].$$

**More Cases:** If

$$v(a_{11}^2\alpha) \leq v(a_{12}^2\beta), \quad v(a_{21}^2\alpha) > v(a_{22}^2\beta),$$

then interchanging  $v_1, v_2$  we come back to Cases I, II.

If

$$v(a_{11}^2\alpha) < v(a_{12}^2\beta), \quad v(a_{21}^2\alpha) \leq v(a_{22}^2\beta),$$

then interchanging also  $v'_1, v'_2$  we come again back to Cases I, II.

Finally, if

$$v(a_{11}^2\alpha) < v(a_{12}^2\beta), \quad v(a_{21}^2\alpha) > v(a_{22}^2\beta),$$

then interchanging  $v_1, v_2$  we come back to Case IV.

Thus Cases I–IV exhaust all possibilities up to interchanging  $v_1, v_2$  and/or  $v'_1, v'_2$ .

This completes Example B.

If the values  $v(\alpha), v(\beta)$  are not square equivalent, then Case IV in Example B does not occur, as observed above. Thus we may state

**Proposition 4.1.** *Assume that  $R$  is a field, and that  $\varphi : R \rightarrow U$  is a tangibly additive supervaluation which is not ghost, and hence is very strong. Let  $q = [\alpha, \beta]$  be a binary diagonal form over  $R$  with  $v(\alpha), v(\beta)$  not square equivalent ( $v := e\varphi$ ). Then all stropicalizations of  $q$  by  $\varphi$  are quasilinear.*

**Remark 4.2.** *This proposition does not contradict Example A. Assume that  $\text{char } R \neq 2$ . If  $q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $q' = [\alpha, \beta]$  are forms over  $R$  with  $\alpha \neq 0, \beta \neq 0$ , and  $q \cong q'$ , then  $\beta = -\lambda^2\alpha$  for some  $\lambda \in R^*$ , and hence  $v(\alpha)$  and  $v(\beta)$  are square equivalent.*

## 5. THE MINIMAL ORDERING ON A FREE $R$ -MODULE

In this section  $R$  is a supertropical semiring. If  $V$  is any module over  $R$ , we define on  $V$  a binary relation  $\leq_V$  as follows:

For any  $x, y \in U$ ,

$$x \leq_V y \iff \exists z \in V : x + z = y. \quad (5.1)$$

This relation is clearly reflexive ( $x \leq x$ ) and transitive ( $x \leq y, y \leq z \Rightarrow x \leq z$ ). It is also antisymmetric, *hence is a partial ordering on the set  $V$* . Indeed, assume that  $x + z = y$  and  $y + w = x$ . This implies  $x + z + w = x$ ,  $y + z + w = y$ , and then

$$x + e(z + w) = x, \quad y + e(z + w) = y.$$

Adding  $z$  at both sides of the first equation, and using that  $z + ez = ez$ , we obtain

$$y = x + e(z + w) = x,$$

as desired.

Clearly, our partial ordering  $\leq_V$  satisfies the rules ( $x, y, z \in V$ )

$$0 \leq z, \quad (5.2)$$

$$x \leq y \Rightarrow x + z \leq y + z. \quad (5.3)$$

(Thus,  $x \leq y, x' \leq y' \Rightarrow x + x' \leq y + y'$ .) It is now obvious that any partial ordering  $\leq'$  on  $V$  with the properties (5.2), (5.3), is a refinement of  $\leq_V$ : If  $x \leq_V y$ , then  $x \leq' y$ .

**Definition 5.1.** *We call  $\leq_V$  the **minimal ordering** on the  $R$ -module  $V$ .<sup>8</sup>*

**Notation 5.2.** *As long as no other orderings of  $V$  come into play, we usually write  $x \leq y$  instead of  $x \leq_V y$ . But notice that if  $W$  is a submodule of  $V$ , it may happen for  $x, y \in W$  that  $x \leq_V y$  but not  $x \leq_W y$ .*

*As usual,  $x < y$  means that  $x \leq y$  and  $x \neq y$ .*

In particular,  $R$  itself carries the minimal ordering  $\leq_R$ . It already showed up in [1, Proposition 11.8] and [4, §5]. Again, we usually write  $\lambda \leq \mu$  instead of  $\lambda \leq_R \mu$ .

Scalar multiplication is compatible with these orderings on  $R$  and  $V$ :

$$\lambda \leq \mu, x \leq y \Rightarrow \lambda x \leq \mu y \quad (5.4)$$

for all  $\lambda, \mu \in \mathbb{R}, x, y \in V$ .

Before moving on to details about minimal orderings, we hasten to point out that these orderings are relevant for the geometry in a supertropical quadratic space. This is apparent already from the definition of quadratic forms [4, Definition 0.1].

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<sup>8</sup>In the special case  $V = R$  the minimal ordering has been discussed already in [4, §5], including an explanation of the term “minimal”.

**Remark 5.3.** As before, let  $V$  be a module over a supertropical semiring  $R$ . If  $(q, b)$  is a quadratic pair on  $V$ , then for all  $x, y, z, w \in V$  the following hold:

$$x \leq_V z \Rightarrow q(x) \leq_R q(z), \quad (5.5)$$

$$x \leq_V z, y \leq_V w \Rightarrow b(x, y) \leq_R b(z, w), \quad (5.6)$$

$$b(x, y) \leq_R q(x + y). \quad (5.7)$$

The minimal ordering of  $R$  has the following detailed description in terms of the  $\nu$ -dominance relation and the sets  $eR$  and  $\mathcal{T} = R \setminus (eR)$ .

**Proposition 5.4.**

- a) Assume that  $x \in eR$ . Then  $x$  is comparable (in the minimal ordering) to every  $y \in R$ . More precisely, using the  $\nu$ -notation,

$$x < y \Leftrightarrow x <_\nu y, \quad (5.8)$$

$$y < x \Leftrightarrow \text{either } y <_\nu x, \text{ or } y \in \mathcal{T} \text{ and } y \cong_\nu x. \quad (5.9)$$

- b) Assume that  $x \in \mathcal{T}, y \in R$ . Then

$$x < y \Leftrightarrow \text{either } x <_\nu y, \text{ or } x \cong_\nu y \text{ and } y \in eR, \quad (5.10)$$

$$y < x \Leftrightarrow y <_\nu x. \quad (5.11)$$

Thus  $x$  and  $y$  are incomparable iff  $y \in \mathcal{T}$  and  $x \neq y$ , but  $x \cong_\nu y$ .

*Proof.* All this can be read off from the description (0.6) of the sum  $x + y$  of  $x, y \in R$  in terms of the  $\nu$ -dominance relation, recalled from [6, §2].<sup>9</sup>  $\square$

From Proposition 5.4 we read off that for any two elements  $x, y$  of  $R$  the maximum  $x \vee y := \max_R \{x, y\}$  exists, namely

$$x \vee y = \begin{cases} x & \text{if } ex < ey; \\ y & \text{if } ex > ey; \\ ex & \text{if } ex = ey. \end{cases} \quad (5.12)$$

Note that

$$e(x \vee y) = (ex) \vee (ey) = ex + ey, \quad (5.13)$$

while for arbitrary  $\lambda \in R$  in general only  $\lambda(x \vee y) \leq (\lambda x) \vee (\lambda y)$ , but here we have equality if  $R$  is a supersemifield.

Assume now that  $V$  is a free  $R$ -module with base  $(\varepsilon_i \mid i \in I)$ . If  $x, y$  are vectors in  $V$  with coordinates  $(x_i \mid i \in I), (y_i \mid i \in I)$ , i.e.,

$$x = \sum_{i \in I} x_i \varepsilon_i \quad y = \sum_{i \in I} y_i \varepsilon_i,$$

where  $x_i \neq 0$  or  $y_i \neq 0$  only for finitely many  $i \in I$ , then clearly

$$x \leq_V y \Leftrightarrow \forall i \in I \quad x_i \leq_R y_i. \quad (5.14)$$

Moreover, the maximum  $x \vee y = \max_V \{x, y\}$ , exists, and

$$x \vee y = \sum_{i \in I} (x_i \vee y_i) \varepsilon_i. \quad (5.15)$$

<sup>9</sup>The general assumption in [6], that the monoid  $(eR, \cdot)$  is cancellative, is not needed here. It is only relevant if products  $xy$  are involved.

It will be helpful below to argue by use of the **support** of an element  $x = \sum_{i \in I} x_i \varepsilon_i$  of the free module  $V$  defined as follows

$$\text{supp}(x) := \{i \in I \mid x_i \neq 0\}. \quad (5.16)$$

As consequence of (5.15) we have

$$\text{supp}(x \vee y) = \text{supp}(x) \cup \text{supp}(y). \quad (5.17)$$

Notice that  $\text{supp}(x)$  is essentially independent of the choice of the base  $(\varepsilon_i \mid i \in I)$ , since up to permutation every other base of  $V$  arises by multiplying the  $\varepsilon_i$  by units of  $R$  [4, Theorem 0.9]. Notice also that  $\text{supp}(x)$  is empty iff  $x = 0$ , and that  $y \leq x$  implies  $\text{supp}(y) \subseteq \text{supp}(x)$ .

## 6. $q$ -MINIMAL VECTORS WITH SMALL SUPPORT

In this section  $R$  is again a supertropical semiring. In all  $R$ -modules we work with their minimal orderings.

**Definition 6.1.**

a) We call a map  $\phi : V \rightarrow W$  between  $R$ -modules  $V, W$  **monotonic** if for any  $x, y \in V$

$$y \leq x \Rightarrow \phi(y) \leq \phi(x).$$

b) Given a monotonic map  $\phi : V \rightarrow W$ , we call a vector  $x \in V$   **$\phi$ -minimal**, if there **does not** exist a vector  $x' < x$  in  $V$  with  $\phi(x') = \phi(x)$ .

**Examples 6.2.**

i) For any  $n \in \mathbb{N}$  and  $c \in R$ , the map  $R \rightarrow R$ ,  $x \mapsto cx^n$ , is additive, and hence monotonic. More generally, every monomial map  $R^n \rightarrow R$ ,

$$(x_1, \dots, x_n) \mapsto cx_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad (\alpha_i \in \mathbb{N}_0),$$

is monotonic, and hence every polynomial map  $f : R^n \rightarrow R$  is monotonic.

ii) Every quadratic form  $q : V \rightarrow R$  on an  $R$ -module  $V$  is monotonic, cf. Remark 5.3. We note the trivial fact that an isotropic vector  $x \in V \setminus \{0\}$  is never  $q$ -minimal, since  $0 < x$ , but  $q(x) = q(0) = 0$ .

Given a quadratic form  $q : V \rightarrow R$ , we turn to the problem of determining the  $q$ -minimal vectors in  $V$  in the case that the  $R$ -module  $V$  is free, and, if possible, at later stages also in more general situations. The following distinction of the vectors in  $V$  will be useful here and elsewhere.

**Definition 6.3.** We call a vector  $x \in V \setminus \{0\}$   **$g$ -isotropic**, if  $q(x) \in eR$ , and we call  $x$   **$g$ -anisotropic**, if  $q(x) \in \mathcal{T}$ .<sup>10</sup> The zero vector is regarded as both  $g$ -isotropic and  $g$ -anisotropic.

**Proposition 6.4.** Assume that  $V$  is free with base  $(\varepsilon_i \mid i \in I)$ . Let  $x \in V \setminus \{0\}$  be  $q$ -minimal. Then  $|\text{supp}(x)| \leq 2$  if  $q(x) \in \mathcal{T}$ , and  $|\text{supp}(x)| \leq 4$  if  $q(x) \in \mathcal{G}$ .

*Proof.* We have a finite non-empty subset  $J = \text{supp}(x)$  of  $I$ , such that  $x = \sum_{i \in J} x_i \varepsilon_i$ , all  $x_i \neq 0$ .

We choose a companion  $b$  of  $q$ . Then

$$q(x) = \sum_{i \in J} x_i^2 q(\varepsilon_i) + \sum_{\substack{i < j \\ i, j \in J}} x_i x_j b(\varepsilon_i, \varepsilon_j). \quad (*)$$

<sup>10</sup>The letter “ $g$ ” alludes to “ghost”.

and  $q(x) \neq 0$ .

If  $q(x) \in \mathcal{T}$ , the sum on the right of  $(*)$  contains a unique  $\nu$ -dominant term. If this term is  $x_k^2 q(\varepsilon_k)$ , then  $x_k \varepsilon_k \leq x$  and  $q(x_k \varepsilon_k) = q(x)$ ; hence  $x = x_k \varepsilon_k$  and  $J = \{k\}$ . If the  $\nu$ -dominant term is  $x_k x_\ell b(\varepsilon_k, \varepsilon_\ell)$ , then  $x_k \varepsilon_k + x_\ell \varepsilon_\ell \leq x$  and again both vectors have the same  $q$ -values, and hence  $x = x_k \varepsilon_k + x_\ell \varepsilon_\ell$ , and  $J = \{k, \ell\}$ . Indeed, then

$$q(x) = x_k x_\ell b(\varepsilon_k, \varepsilon_\ell) \leq q(x_k \varepsilon_k + x_\ell \varepsilon_\ell) \leq q(x).$$

If  $q(x) \in \mathcal{G}$ , then on the right of  $(*)$  there exists either a  $\nu$ -dominant term, which is ghost, or there exist two  $\nu$ -dominant terms which are tangible. In the first case, we see as above that  $|J| \leq 2$ , and in the second that  $|J| \leq 4$ .  $\square$

**Corollary 6.5.** *Assume in Proposition 6.4 also that  $q$  is quasilinear. Then  $|\text{supp}(x)| = 1$  if  $q(x) \in \mathcal{T}$ , and  $|\text{supp}(x)| \leq 2$  if  $q(x) \in \mathcal{G}$ .*

*Proof.* We choose the companion  $b = 0$ . Now, in the above arguments no  $\nu$ -dominant terms  $x_k x_\ell b(\varepsilon_k, \varepsilon_\ell)$  show up.  $\square$

Recall from the last lines of §5 that for vectors  $x', x$  in  $V$  with  $x' \leq x$  the support of  $x'$  is contained in the support of  $x$ . Thus in searching for  $q$ -minimal vectors in  $V$  it is not loss of generality to assume that  $|I| \leq 4$ . If  $q$  is quasilinear we may even assume that  $|I| \leq 2$ .

We now deal with the case that  $|I| \leq 2$ , postponing the cases  $|I| = 3$  and  $|I| = 4$  to the next section.

**Proposition 6.6.**

- a) *Assume that  $V$  is free with a single base vector  $\varepsilon_1$ . When  $q(\varepsilon_1) \in \mathcal{T}$ , all vectors in  $V$  are  $q$ -minimal. If  $q(\varepsilon_1) \in \mathcal{G}$ , a vector  $\lambda \varepsilon_1$  is  $q$ -minimal iff  $\lambda \in \mathcal{T}$ .*
- b) *Assume that  $V$  is free with base  $(\varepsilon_1, \varepsilon_2)$ , and that  $q$  is quasilinear. Let  $\alpha_1 := q(\varepsilon_1)$ ,  $\alpha_2 := q(\varepsilon_2)$ . A vector  $x = \lambda \varepsilon_1 + \mu \varepsilon_2$  with  $\lambda, \mu \neq 0$  is  $q$ -minimal iff  $\lambda, \mu, \alpha_1, \alpha_2 \in \mathcal{T}$  and  $\lambda^2 \alpha_1 \cong_\nu \mu^2 \alpha_2$ . (Thus every  $q$ -minimal vector with  $|\text{supp}(x)| = 2$  is  $g$ -isotropic.)*

*Proof.* a): Let  $\alpha_1 := q(\varepsilon_1)$  and  $x := \lambda \varepsilon_1 \in V$ . We have  $q(x) = \lambda^2 \alpha_1$ . Assume first that  $\alpha_1 \in \mathcal{T}$ . If  $x' = \lambda' \varepsilon_1$  is a second vector, then  $x' < x$  iff  $\lambda' < \lambda$  iff  $\lambda'^2 \alpha_1 < \lambda^2 \alpha_1$ . Thus  $x$  is  $q$ -minimal. Assume now that  $\alpha_1 \in \mathcal{G}$ . If  $\lambda \in \mathcal{G}$ , there exists  $\lambda' \in \mathcal{T}$  with  $\lambda' \cong_\nu \lambda$ , and then  $\lambda' < \lambda$ . For  $x' = \lambda' \varepsilon_1$  we have  $x' < x$ , but  $q(x') = \lambda'^2 \alpha_1 = \lambda^2 \alpha_1 = q(x)$ . Thus  $x$  is not  $q$ -minimal. If  $\lambda \in \mathcal{T}$  and  $\lambda' < \lambda$  then  $\lambda' <_\nu \lambda$  (cf. (5.11)); hence

$$q(x') = \lambda'^2 \alpha_1 <_\nu \lambda^2 \alpha_1 = q(x),$$

and a fortiori  $q(x') < q(x)$ . Thus  $x$  is  $q$ -minimal.

b): We have  $q(x) = \lambda^2 \alpha_1 + \mu^2 \alpha_2$ . If  $q(x) = 0$ , then  $x$  is not  $q$ -minimal (cf. Example 6.2.ii).

Assume now that  $q(x) \neq 0$ . If  $\lambda^2 \alpha_1 <_\nu \mu^2 \alpha_2$  then  $q(x) = \mu^2 \alpha_2 = q(\mu \varepsilon_2)$ , and  $x$  is not  $q$ -minimal, ditto if  $\lambda^2 \alpha_1 >_\nu \mu^2 \alpha_2$ . Assume henceforth that  $\lambda^2 \alpha_1 \cong_\nu \mu^2 \alpha_2$ . Then  $q(x) \in \mathcal{G}$  and  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$ . If  $\lambda^2 \alpha_1$  or  $\mu^2 \alpha_2$  is ghost, then  $q(x) = q(\lambda \varepsilon_1)$ , resp.  $q(x) = q(\mu \varepsilon_2)$ , and thus  $x$  is not  $q$ -minimal. We are left with the case that both  $\lambda^2 \alpha_1$ ,  $\mu^2 \alpha_2$  are tangible. This means that  $\lambda, \mu, \alpha_1, \alpha_2 \in \mathcal{T}$ .

If  $x' < x$ , then either  $x' \leq \lambda' \varepsilon_1 + \mu \varepsilon_2$  or  $x' \leq \lambda \varepsilon_1 + \mu' \varepsilon_2$  with  $\lambda' < \lambda$ , resp.  $\mu' < \mu$ . In the first case,  $\lambda' <_\nu \lambda$  (cf. (5.11)), hence  $\lambda'^2 \alpha_1 <_\nu \lambda^2 \alpha_1 \cong_\nu \mu^2 \alpha_2$ , and

$$q(x') \leq q(\lambda' \varepsilon_1 + \mu \varepsilon_2) = \mu^2 \alpha_2 < \mu^2 \alpha_2 = q(x).$$

In the second case,  $q(x') < q(x)$  for the same reason. Thus  $x$  is  $q$ -minimal.  $\square$

We now assume that  $\mathcal{G}$  is a cancellative monoid under multiplication and  $\mathcal{G} = e\mathcal{T}$ , furthermore that  $(q, b)$  is a quadratic pair on the free binary module  $V := R\varepsilon_1 + R\varepsilon_2$ . We search for all  $q$ -minimal vectors in  $V$  with full support.

Let  $\alpha_1 := q(\varepsilon_1)$ ,  $\alpha_2 := q(\varepsilon_2)$ ,  $\beta := b(\varepsilon_1, \varepsilon_2)$ , and  $x = x_1\varepsilon_1 + x_2\varepsilon_2$  with  $x_1 \neq 0$ ,  $x_2 \neq 0$ . Then

$$q(x) = \alpha_1 x_1^2 + \beta x_1 x_2 + \alpha_2 x_2^2. \quad (**)$$

Looking at the  $\nu$ -dominant terms in the sum  $(**)$  we will run through several cases and will easily find out when  $x$  is  $q$ -minimal.

- 0) Assume that  $\alpha_1 x_1^2$  (or  $\alpha_2 x_2^2$ ) is the only  $\nu$ -dominant term. Then  $q(x) = q(x_1\varepsilon_1)$  or  $q(x) = q(x_2\varepsilon_2)$ . Clearly  $x$  is not  $q$ -minimal.
- 1) Assume that both  $\alpha_1 x_1^2$  and  $\alpha_2 x_2^2$  are  $\nu$ -dominant. If, say,  $\alpha_1 x_1^2$  is ghost, then  $q(x) = q(x_1\varepsilon_1)$  again, and  $x$  is not  $q$ -minimal. If both  $\alpha_1 x_1^2$  and  $\alpha_2 x_2^2$  are tangible, then for a vector  $x' = x'_1\varepsilon_1 + x'_2\varepsilon_2 < x$  either  $x'_1 < x_1$  or  $x'_2 < x_2$ , which implies  $x'_1 <_\nu x_1$  or  $x'_2 <_\nu x_2$ , since both  $x'_1, x'_2$  are tangible. We conclude that  $q(x') < q(x)$ . Thus  $x$  is  $q$ -minimal iff  $\alpha_1, \alpha_2, x_1, x_2$  are all tangible.
- 2) Assume that  $\alpha_1 x_1^2 \cong_\nu \beta x_1 x_2 > \alpha_2 x_2^2$ . Then  $q(x) = e\alpha_1 x_1^2 = e\beta x_1 x_2 \in \mathcal{G}$ . If  $\alpha_1 x_1^2 \in \mathcal{G}$ , then choosing  $x'_1 \in \mathcal{T}$  with  $ex'_1 = x_1$  we obtain a vector  $x' = x'_1\varepsilon_1 + x_2\varepsilon_2 < x$  with  $q(x') = \alpha'_1 x_1^2 + \beta x'_1 x_2 = q(x)$ , and so  $x$  is not  $q$ -minimal.  
Assume now that  $\alpha_1 x_1^2 \in \mathcal{T}$ . If  $x' = x'_1\varepsilon_1 + x'_2\varepsilon_2 < x$ , then either  $x'_1 < x_1$ ,  $x'_2 \leq x_2$ , or  $x'_1 = x_1$ ,  $x'_2 < x_2$ . If  $x'_1 < x_1$ , then  $x'_1 <_\nu x_1$ , whence  $\alpha_1 x_1'^2 <_\nu \alpha_1 x_1^2$ ,  $\beta x'_1 x_2 <_\nu \beta x_1 x_2$ , and we see that  $q(x') < q(x)$ . But if  $x'_1 = x_1$ ,  $x'_2 < x_2$ ,  $ex'_2 = x_2$ , and  $\beta \in \mathcal{G}$ , then  $q(x') = q(x)$ , while if  $\beta \in \mathcal{T}$  this cannot happen. We conclude that  $x$  is  $q$ -minimal iff  $\alpha_1, \beta, x_1$  are all tangible.
- 3) Analogously, if  $\alpha_2 x_2^2 \cong_\nu \beta x_1 x_2 > \alpha_1 x_1^2$ , then  $x$  is  $q$ -minimal iff  $\alpha_2, \beta, x_2$  are all tangible.
- 4) Assume that  $\alpha_1 x_1^2 <_\nu \beta x_1 x_2$ ,  $\alpha_2 x_2^2 <_\nu \beta x_1 x_2$ . Now  $q(x) = \beta x_1 x_2$ . Arguing similarly as in Case 3), we see that, when  $\beta \in \mathcal{G}$  then  $x$  is  $q$ -minimal iff  $x_1 \in \mathcal{T}$  and  $x_2 \in \mathcal{T}$ , while when  $\beta \in \mathcal{T}$ , then  $x$  is  $q$ -minimal iff  $x_1 \in \mathcal{T}$  or  $x_2 \in \mathcal{T}$ . Thus all together  $x$  is  $q$ -minimal iff at most one of the elements  $\beta, x_1, x_2$  is ghost.

Summarizing we obtain

**Theorem 6.7.** *Assume that  $V$  is free with base  $\varepsilon_1, \varepsilon_2$  and  $x = x_1\varepsilon_1 + x_2\varepsilon_2$  with  $x_1 \neq 0$ ,  $x_2 \neq 0$ . Let  $q = \begin{bmatrix} \alpha_1 & \beta \\ & \alpha_2 \end{bmatrix}$ . Then  $x$  is  $q$ -minimal exactly in the following cases:*

- 1)  $\alpha_1 x_1^2 \cong_\nu \alpha_2 x_2^2 \geq_\nu \beta x_1 x_2$  and  $\alpha_1, \alpha_2, x_1, x_2 \in \mathcal{T}$ ;
- 2)  $\alpha_1 x_1^2 \cong_\nu \beta x_1 x_2 >_\nu \alpha_2 x_2^2$  and  $\alpha_1, \beta, x_1 \in \mathcal{T}$ ;
- 3)  $\alpha_2 x_2^2 \cong_\nu \beta x_1 x_2 >_\nu \alpha_1 x_1^2$  and  $\alpha_2, \beta, x_2 \in \mathcal{T}$ ;
- 4)  $\beta x_1 x_2 >_\nu \alpha_1 x_1^2 + \alpha_2 x_2^2$  and at most one of the elements  $\beta, x_1, x_2$  is ghost.

**Comment 6.8.** *In order to clarify the situation observe that in Cases 2)-4) we have  $\alpha_1 x_1^2 \cdot \alpha_2 x_2^2 <_\nu (\beta x_1 x_2)^2$ , whence  $\alpha_1 \alpha_2 <_\nu \beta^2$ , while in Case 1) we have  $\beta^2 <_\nu \alpha_1 \alpha_2$ . Thus  $(\varepsilon_1, \varepsilon_2)$  is excessive w.r. to  $q$  in Cases 2)-4), but quasilinear in Case 1).*

Concerning  $g$ -anisotropic vectors we note the following immediate consequence of Theorem 6.7.

**Corollary 6.9.** *Assume again that  $x = x_1\varepsilon_1 + x_2\varepsilon_2$  and  $q = \begin{bmatrix} \alpha_1 & \beta \\ & \alpha_2 \end{bmatrix}$ . Then  $x$  is  $q$ -minimal and  $g$ -anisotropic iff  $\beta, x_1, x_2$  are tangible and  $\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2 <_\nu \beta x_1 x_2$ .*



Returning to the tables of  $q$ -values in §3 it is of interest to ask which of the vectors  $\lambda\varepsilon_1 + \mu\varepsilon_2$  there are  $q$ -minimal. We only consider the case that  $\alpha^2 >_\nu \alpha_1\alpha_2$  in the notations used there, since otherwise  $q$  is quasilinear and the matter is settled by Proposition 6.6.b.

**Theorem 6.10.** *Assume that  $R$  is a nontrivial tangible supersemifield, and  $q$  is a quadratic form on the free binary  $R$ -module  $V = R\varepsilon_1 + R\varepsilon_2$ . Let  $b$  be a companion of  $q$ , and assume that  $\alpha_1\alpha_2 <_\nu \alpha^2$  with  $\alpha_1 := q(\varepsilon_1)$ ,  $\alpha_2 := q(\varepsilon_2)$ ,  $\alpha := b(\varepsilon_1, \varepsilon_2)$ . We use Notations 3.1 and 3.6. Let  $x = \lambda\varepsilon_1 + \mu\varepsilon_2$  with  $\lambda, \mu \neq 0$ .*

- i) *If  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$ , then  $x$  is  $q$ -minimal iff either  $\lambda \cong_\nu \zeta\mu$  and  $\alpha_1, \lambda \in \mathcal{T}$ , or  $\lambda \cong_\nu \eta\mu$  and  $\alpha_2, \mu \in \mathcal{T}$ , or  $\eta\mu <_\nu \lambda <_\nu \zeta\mu$  and at most one of the three elements  $\alpha, \lambda, \mu$  is ghost*
- ii) *If  $\alpha_1 \neq 0$ ,  $\alpha_2 = 0$ , then  $x$  is  $q$ -minimal iff either  $\lambda \cong_\nu \zeta\mu$  and  $\alpha_1, \lambda \in \mathcal{T}$ , or  $\lambda <_\nu \zeta\mu$  and at most one of the elements  $\alpha, \lambda$  is ghost.*
- iii) *If  $\alpha_1 = \alpha_2 = 0$ , then  $x$  is  $q$ -minimal iff at most one of the elements  $\alpha, \lambda, \mu$  is ghost.*

*Proof.* Browse through tables (3.7), (3.8), (3.11), (3.12) and apply Theorem 6.7, reading  $\lambda, \mu, \alpha$  for  $x_1, x_2, \beta$ .  $\square$

## 7. $q$ -MINIMAL VECTORS WITH BIG SUPPORT

Again we assume that  $R$  is a tangible supertropical semiring,  $\mathcal{G}$  is a cancellative monoid,  $V$  is a free  $R$ -module with base  $(\varepsilon_i \mid i \in I)$ , and  $q : V \rightarrow R$  is a quadratic form. For later use, we adopt the following notation.

**Notation 7.1.** *Let  $x = \sum_{i \in I} x_i \varepsilon_i \in V$  and  $J$  a subset of  $I$ . We put*

$$x(J) := \sum_{i \in J} x_i \varepsilon_i.$$

*If  $J = \{i\}$  or  $J = \{i, j\}$ ,  $i \neq j$ , we write for short  $x(i)$  or  $x(i, j)$  instead of  $x(\{i\})$ ,  $x(\{i, j\})$ .*

Assume now that  $I = \{1, \dots, n\}$  with  $n = 3$  or  $n = 4$ , and that  $x \in V$  is a vector of full support,

$$x = \sum_{i=1}^n x_i \varepsilon_i, \quad \text{all } x_i \neq 0.$$

We choose a companion  $b$  of  $q$ , and then have a presentation

$$q(x) = \sum_{i=1}^n \alpha_i x_i^2 + \sum_{i < j} \beta_{ij} x_i x_j. \quad (7.1)$$

We ask, under which conditions is  $x$   $q$ -minimal, and then search for possibilities to write  $x$  as the supremum  $y \vee z$  of two  $q$ -minimal vectors  $y, z \in V$  of small support, i.e.,  $|\text{supp}(y)| \leq 2$ ,  $|\text{supp}(z)| \leq 2$ .

As in §6, we look for the  $\nu$ -dominant terms in the sum (7.1). If there is only one dominant term,  $\alpha_i x_i^2$  or  $\beta_{ij} x_i x_j$ , then  $q(x) = q(x(i))$  or  $q(x) = q(x(i, j))$ , and so  $x$  is not  $q$ -minimal. Henceforth, we assume always that there are at least two dominant terms, and so  $q(x) \in \mathcal{G}$ . Furthermore, we assume that all  $\nu$ -dominant terms are tangible, since otherwise again  $q(x) = q(x(J))$  for some  $J \subsetneq I$ .

We first study the case  $n = 3$  and run through several subcases, as follows:

- A) Assume that in (7.1) there occurs a  $\nu$ -dominant term  $\alpha_i x_i^2$ . Then, if  $x$  is  $q$ -minimal there is exactly one further dominant terms  $\beta_{jk} x_j x_k$  and  $(i, j, k)$  is a permutation of  $(1, 2, 3)$ , since otherwise again  $q(x) = q(x(J))$  for some  $J \subsetneq I$ . We have

$$x = x(i) \vee x(j, k),$$

$$\text{and } q(x(i)) = \alpha_i x_i^2 \in \mathcal{T},$$

$$q(x(j, k)) = \alpha_j x_j^2 + \beta_{ik} x_j x_k + \alpha_k x_k^2 \in \mathcal{T}.$$

It follows that

$$\alpha_j x_j^2 + \alpha_k x_k^2 <_\nu \beta_{jk} x_j x_k,$$

and we read off from Theorem 6.7 that  $x(j, k)$  is  $q$ -minimal. By Proposition 6.6.a also  $x(i)$  is  $q$ -minimal.

Note furthermore that

$$b(x(i), x(j, k)) <_\nu q(x(i)) \cong_\nu q(x).$$

Assume now that all  $\nu$ -dominant terms in the sum (7.1) are of the form  $\beta_{ij} x_i x_j$ . We distinguish two subcases.

- B) Exactly two of the terms  $\beta_{ij} x_i x_j$  are  $\nu$ -dominant.  
 C) All three such terms are  $\nu$ -dominant.

In Case B there is a permutation  $(i, j, k)$  of  $(1, 2, 3)$  such that

$$q(x) \cong_\nu \beta_{ij} x_i x_j \cong_\nu \beta_{ik} x_i x_k >_\nu \beta_{jk} x_j x_k, \quad (7.2)$$

while in Case C we have

$$q(x) \cong_\nu \beta_{12} x_1 x_2 \cong_\nu \beta_{13} x_1 x_3 \cong_\nu \beta_{23} x_2 x_3. \quad (7.3)$$

In both cases  $q(x) >_\gamma \alpha_i x_i^2$  for all  $i \in I$ . It follows by Corollary 6.9 that in Case B both vectors  $x(i, j)$  and  $x(i, k)$  are  $g$ -anisotropic and  $q$ -minimal, while in Case C all three vectors  $x(1, 2)$ ,  $x(1, 3)$ ,  $x(2, 3)$  have these properties. Due to our knowledge of all  $\nu$ -dominant terms in the sum (7.1), we see that in Case B

$$b(x(j), x(k)) <_\nu q(x(i, j)) \cong_\nu q(x(i, k)) \cong_\nu q(x),$$

while in Case C for every 2-element subset  $\{r, s\}$  of  $I$  we have  $b(x(r), x(s)) \in \mathcal{T}$  and

$$b(x(r), x(s)) \cong_\nu q(x(r, s)) \cong_\nu q(x).$$

{Observe that  $b(\varepsilon_i, \varepsilon_i) \leq_\nu \alpha_i$ , cf. [4, Ineq. (1.9)].}

- D) We turn to the case  $n = 4$ , which is easier. Assume that  $x$  is  $q$ -minimal. Then we have exactly two  $\nu$ -dominant terms in the sum (7.1),  $\beta_{ij} x_i x_j$ ,  $\beta_{\ell k} x_\ell x_k$ , with  $\{i, j\}$  disjoint from  $\{k, \ell\}$ , since otherwise there would exist a set  $S \subsetneq I$  with  $q(x(S)) = q(x)$ . Moreover, these terms are tangible.

Arguing as above we conclude easily that there is a partition  $I = J \dot{\cup} K$  with  $|J| = |K| = 2$ , such that  $x(J)$  and  $x(K)$  are  $g$ -anisotropic and  $q$ -minimal with

$$q(x(J)) \cong_\nu q(x(K)) \cong_\nu q(x),$$

while  $q(x(S)) <_\nu q(x)$  for all other subsets  $S$  of  $I$  with  $|S| \leq 2$ . Also for any two different subsets  $S, T$  of  $I$  with  $|S| \leq 2$ ,  $|T| \leq 2$ , including  $S = J$ ,  $T = K$ , we have

$$b(x(S), x(T)) <_\nu q(x).$$

Summarizing the essentials of this study, we obtain

**Theorem 7.2.** *Assume that  $x$  is  $q$ -minimal and  $\text{supp}(x) = I = \{1, \dots, n\}$  with  $n \geq 3$ . Then  $x$  is  $g$ -isotropic and exactly one of the following four cases holds:*

- A)  $n = 3$ . *There is a unique partition  $I = J \dot{\cup} K$  with  $|J| = 1$ ,  $|K| = 2$ , both  $x(J)$ ,  $x(K)$   $g$ -anisotropic and  $q$ -minimal, and  $q(x(J)) \cong_\nu q(x(K)) \cong_\nu q(x)$ .*
- B)  $n = 3$ . *There are exactly two 2-element subsets  $J$  and  $K$  of  $I$  with  $x(J)$ ,  $x(K)$   $g$ -anisotropic and  $q$ -minimal and  $q(x(J)) \cong_\nu q(x(K)) \cong_\nu q(x)$ .*
- C)  $n = 3$ . *For any 2-element subset  $J$  of  $I$ , the vector  $x(J)$  is  $q$ -minimal and  $g$ -anisotropic and  $q(x(J)) \cong_\nu q(x)$ . Thus the properties listed in B) hold for any two 2-element subsets  $J, K$  of  $I$ .*
- D)  $n = 4$ . *There are exactly two 2-element subsets  $J$  and  $K$  of  $I$  such that  $x(J)$ ,  $x(K)$  are  $g$ -anisotropic,  $q$ -minimal and*

$$q(x(J)) \cong_\nu q(x(K)) \cong_\nu q(x).$$

*$J$  and  $K$  are disjoint.*

*In all four cases, we have  $I = J \cup K$ , whence  $x = x(J) \vee x(K)$  for the sets  $J, K$  from above. Moreover, in Cases A and D,*

$$b(x(J), x(K)) <_\nu q(x). \quad (7.4)$$

*In Case B,*

$$b(x(J), x(K)) = q(x), \quad (7.5)$$

*whereas*

$$b(x(J \setminus K), x(K \setminus J)) \cong_\nu q(x). \quad (7.6)$$

*In Case C, (7.5) holds for any two different 2-element subsets  $J, K$  of  $I$ , and moreover*

$$b(x(J \setminus K), x(K \setminus J)) \cong_\nu q(x), \quad b(x(J \setminus K), x(K \setminus J)) \in \mathcal{T}. \quad (7.7)$$

As before we assume that  $V$  is free with base  $(\varepsilon_i \mid i \in I)$ ,  $I = \{1, \dots, n\}$ ,  $n = 3$  or  $4$ . Given two  $g$ -anisotropic  $q$ -minimal vectors  $y, z \in V$  of small support, we now ask for conditions under which the vector  $x := y \vee z$  is  $q$ -minimal and has full support  $I$ . In view of Theorem 7.2, we will be content to assume from the beginning that

$$b(y, z) \leq_\nu q(y) \cong_\nu q(z). \quad (7.8)$$

A satisfactory converse to Theorem 7.2 in the cases A) and B) runs as follows.

**Theorem 7.3.** *Assume that  $y, z \in V$  are  $g$ -anisotropic and  $q$ -minimal, and furthermore that  $y \vee z$  has full support  $I$ , and*

$$b(y, z) <_\nu q(y) \cong_\nu q(z). \quad (7.9)$$

*Assume finally that  $n = 3$ ,  $|\text{supp}(y)| = 1$ ,  $|\text{supp}(z)| = 2$ , or  $n = 4$ , and  $|\text{supp}(y)| = |\text{supp}(z)| = 2$ . Then  $x := y \vee z$  is  $q$ -minimal.*

*Proof.* We have  $\text{supp}(y) \cup \text{supp}(z) = I$ , which forces  $\text{supp}(y) \cap \text{supp}(z) = \emptyset$ .

a) Assume first that  $n = 3$ . After a permutation of the  $\varepsilon_i$ , we may assume

$$y = y_1 \varepsilon_1, \quad z = z_2 \varepsilon_2 + z_3 \varepsilon_3,$$

and then have  $x = \sum_{i=1}^3 x_i \varepsilon_i$  with

$$x_1 = y_1, \quad x_2 = z_2, \quad x_3 = z_3.$$

It follows from Proposition 6.6.a and Corollary 6.9 that  $\alpha_1 x_1^2 = q(y) \in \mathcal{T}$  and

$$\alpha_2 x_2^2 + \alpha_3 x_3^2 <_\nu \beta_{23} x_2 x_3 = q(z) \in \mathcal{T}. \quad (7.10)$$

Thus  $x_1, x_2, x_3, \alpha_1, \beta_{23}$  are all tangible. Further by assumption (7.9)

$$\beta_{11} x_1^2 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 <_\nu \alpha_1 x_1^2 \cong_\nu \beta_{23} x_2 x_3. \quad (7.11)$$

Here  $\beta_{11} = b(\varepsilon_1, \varepsilon_1) \leq_\nu \alpha$  (cf. [4, Ineq. (1.9)]). It follows that

$$q(x) = \alpha_1 x_1^2 + \beta_{23} x_2 x_3 = eq(y) = eq(z).$$

Given  $x' = \sum_1^3 x'_i \varepsilon_i < x$ , we want to prove that  $q(x') < q(x)$ . It suffices to consider the case  $x'_1 < x_1$ ,  $x'_2 = x_2$ ,  $x'_3 = x_3$  and  $x'_1 = x_1$ ,  $x'_2 < x_2$ ,  $x'_3 = x_3$ . Notice that  $x'_i < x_i$  implies  $x'_i <_\nu x_i$  since all  $x_i$  are tangible.

In the first case  $\beta_{23} x'_2 x'_3 = \beta_{23} x_2 x_3$ , and we learn from (7.10) and (7.11) that in the sum

$$\sum_1^3 \alpha_i x_i'^2 + \sum_{i < j} \beta_{ij} x'_i x'_j = q(x')$$

there is only one  $\nu$ -dominant term  $\beta_{23} x_2 x_3$ , which is tangible. Thus

$$q(x') = \beta_{23} x_2 x_3 \in \mathcal{T}, \quad \text{and} \quad q(x') \cong_\nu q(x).$$

Since  $q(x)$  is ghost, this implies  $q(x') < q(x)$ . In the second case where  $x'_2 < x_2$ , we can argue in the same way, now obtaining  $q(x') = \alpha_1 x_2^2 \in \mathcal{T}$  and then  $q(x') < q(x)$ . Thus  $x$  is indeed  $q$ -minimal.

b) Now let  $n = 4$ . We may assume that  $\text{supp}(y) = \{1, 2\}$  and  $\text{supp}(z) = \{3, 4\}$ , whence

$$y = y_1 \varepsilon_1 + y_2 \varepsilon_2, \quad z = z_3 \varepsilon_3 + z_4 \varepsilon_4,$$

and  $x = \sum_1^4 x_i \varepsilon_i$  with

$$x_1 = y_1, \quad x_2 = y_1, \quad x_3 = z_3, \quad x_4 = z_4.$$

Trivially  $y = x(1, 2)$ ,  $z = x(3, 4)$ . We infer from Corollary 6.9 that

$$\alpha_1 x_1^2 + \alpha_2 x_2^2 <_\nu \beta_{12} x_1 x_2 = q(y) \in \mathcal{T}, \quad (7.12)$$

$$\alpha_3 x_3^2 + \alpha_4 x_4^2 <_\nu \beta_{34} x_3 x_4 = q(z) \in \mathcal{T}, \quad (7.13)$$

and further from Condition (7.7) that

$$\beta_{13} x_1 x_3 + \beta_{14} x_1 x_4 + \beta_{23} x_2 x_3 + \beta_{24} x_2 x_4 <_\nu q(y) \cong_\nu q(z).$$

Let  $x' < x$ , and assume w.l.o.g. that exactly one coordinate  $x'_i < x_i$ , say  $x'_1 < x_1$ , which implies  $x'_1 <_\nu x_1$ . If  $q(x') = q(x)$  would hold, then

$$q(x') = \beta_{12} x'_1 x_2 + \beta_{34} x_3 x_4 = \beta_{34} x_3 x_4.$$

But  $q(x')$  is tangible, while  $q(x)$  is ghost. This contraction proves that  $q(x') < q(x)$ , and we conclude that  $x$  is  $q$ -minimal.  $\square$

If  $n = 3$  and  $|\text{supp}(y)| = |\text{supp}(z)| = z$ , then a crude converse to Theorem 7.2, analogous to Theorem 7.3 with only condition (7.9) replaced by (7.8), does not hold, as the following example shows.

**Example 7.4.** Let  $y = y_1\varepsilon_1 + y_2\varepsilon_2$  and  $z = z_1\varepsilon_1 + z_3\varepsilon_3$  with  $y_1, y_2, z_1, z_3 \in \mathcal{T}$  and  $ey_1 = ez_1$ ,  $ey_2 = ez_3$ , but  $y_1 \neq z_1$ . Then

$$x := y \vee z = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3$$

with

$$x_1 = ey_1, \quad x_2 = y_2, \quad x_3 = z_3.$$

Assume further that

- 1)  $\beta_{12}, \beta_{13} \in \mathcal{T}$ ,
- 2)  $\alpha_1 y_1^2 + \alpha_2 y_2^2 <_\nu \beta_{12} y_1 y_2 \in \mathcal{T}$ ,
- 3)  $\alpha_1 z_1^2 + \alpha_3 z_3^2 <_\nu \beta_{13} z_1 z_3$ .

Both  $y$  and  $z$  are  $q$ -minimal and  $g$ -anisotropic by Corollary 6.9, and

$$q(y) = \beta_{12} y_1 y_2 \cong_\nu \beta_{13} z_1 z_3 = q(z).$$

Since  $\beta_{11} := b(\varepsilon_1, \varepsilon_1) \leq_\nu \alpha_1$  and  $ey_1 = ez_1$ , we have

$$b(y_1\varepsilon_1, z_1\varepsilon_1) \leq_\nu \alpha_1 y_1^2 \cong_\nu \alpha_1 z_1^2$$

and conclude that

$$b(y, z) = \beta_{11} y_1 z_1 + \beta_{12} z_1 y_2 + \beta_{13} y_1 z_3 = eq(y) = eq(z).$$

Thus Condition (7.8) is valid. We have  $x = y + z$ , whence

$$q(x) = q(y) + q(z) + b(y, z) = eq(y).$$

Let now  $x' := y_1\varepsilon_1 + y_2\varepsilon_2 + z_3\varepsilon_3$ . Then  $x' < x$ , but

$$q(x') \geq \beta_{12} y_1 y_2 + \beta_{13} y_1 z_3 = eq(y).$$

Thus  $q(x') = q(x)$ . This proves that  $x$  is not  $q$ -minimal.

The vector  $x = y \vee z$  in Theorem 7.3 obviously satisfies  $y = x(J)$ ,  $z = x(K)$  with  $J := \text{supp}(y)$ ,  $K := \text{supp}(z)$ , while for the vector  $y \vee z$  in Example 7.4 this does not hold. If we insist on the property  $y = x(J)$ ,  $z = x(K)$ , then we obtain a converse of Theorem 7.2 also in the cases B) and D) as follows.

**Theorem 7.5.** Let  $n = 3$ . Assume that  $y, z \in V$  are  $g$ -anisotropic and  $q$ -minimal with respective support  $J, K$  such that  $|J| = 2$ ,  $|K| = 2$ ,  $J \cup K = I$ , whence  $J \cap K$  is a singleton. Assume that  $y(J \cap K) = z(J \cap K)$  and furthermore that either

$$b(y(J \setminus K), z(K \setminus J)) <_\nu q(y) \cong_\nu q(z); \quad (7.14)$$

or

$$b(y(J \setminus K), z(K \setminus J)) \in \mathcal{T}, \quad b(y(J \setminus K), z(K \setminus J)) \cong_\nu q(y) \cong_\nu q(z). \quad (7.15)$$

Then  $x := y \vee z$  is  $q$ -minimal and, of course,  $x(J) = y$ ,  $x(K) = z$ .

*Proof.* We may assume that  $J = \{1, 2\}$ ,  $K = \{1, 3\}$ , and then have

$$y = y_1\varepsilon_1 + y_2\varepsilon_2, \quad z = z_1\varepsilon_1 + z_3\varepsilon_3$$

with  $y_1 = z_1$ . Then  $x = \sum_{i=1}^3 x_i\varepsilon_i$  with

$$x_1 = y_1 = z_1, \quad x_2 = y_2, \quad x_3 = z_3.$$

It follows from Corollary 6.9 that

$$(1) \quad \alpha_1 x_1^2 + \alpha_2 x_2^2 <_\nu \beta_{12} x_1 x_2 = q(y) \in \mathcal{T},$$

$$(2) \alpha_1 x_1^2 + \alpha_3 x_3^2 <_\nu \beta_{13} x_1 x_3 = q(z) \in \mathcal{T}.$$

Assume that  $x' = \sum_1^3 x'_i \varepsilon_i$  is given with either

$$\begin{aligned} x'_1 < x_1, \quad x'_2 = x_2, \quad x'_3 = x_3 \quad \text{or} \\ x'_1 = x_1, \quad x'_2 < x_2, \quad x'_3 = x_3. \end{aligned}$$

We will prove that  $q(x') < q(x)$ , and then will be done.

Taking into account that

$$b(y(J \setminus K), z(K \setminus J)) = b(y_2 \varepsilon_2, z_3 \varepsilon_3) = \beta_{23} x_2 x_3,$$

we see that

$$(3) \beta_{23} x_2 x_3 <_\nu \beta_{12} x_1 x_2 \cong_\nu \beta_{13} x_1 x_3,$$

while (7.15) says that

$$(4) \beta_{23} x_2 x_3 \in \mathcal{T}, \quad \beta_{23} x_2 x_3 \cong_\nu \beta_{12} x_1 x_2 \cong_\nu \beta_{13} x_1 x_3.$$

Assume that (3) holds. If  $x'_1 < x_1$ , then  $x'_1 <_\nu x_1$ , and thus

$$\beta_{12} x'_1 x_2 <_\nu \beta_{12} x_1 x_2, \quad \beta_{13} x'_1 x_3 <_\nu \beta_{13} x_1 x_3.$$

It follows from (1), (2), (3) that  $q(x') <_\nu q(x)$ , whence  $q(x') < q(x)$ . If  $x'_2 < x_2$ , then  $x'_2 <_\nu x_2$ , and thus

$$\beta_{12} x_1 x'_2 <_\nu \beta_{12} x_1 x_2, \quad \beta_{23} x'_2 x_3 <_\nu \beta_{23} x_2 x_3.$$

Now we conclude from (1), (2), (3) that

$$q(x') = \beta_{13} x_1 x_3 \cong_\nu q(x).$$

But  $q(x') \in \mathcal{T}$ ,  $q(x) \in \mathcal{G}$ , and so  $q(x') < q(x)$  again.

Assume finally that (4) holds. If  $x'_1 < x_1$ , we see by the same reasoning that

$$q(x') = \beta_{23} x_2 x_3 \cong_\nu q(x),$$

while if  $x'_2 < x_2$  then

$$q(x') = \beta_{13} x_1 x_3 \cong_\nu q(x).$$

In both cases  $q(x') \in \mathcal{T}$ ,  $q(x) \in \mathcal{G}$ , and so  $q(x') < q(x)$ . This completes the proof that  $x$  is  $q$ -minimal.  $\square$

We complement Theorems 7.2, 7.3, 7.5 by an observation on certain pairs of  $q$ -minimal vectors.

**Theorem 7.6.** *Assume that  $x, y \in V$  are  $q$ -minimal vectors with  $y < x$  and  $q(y) \cong_\nu q(x)$ . Let  $J := \text{supp}(y)$ . Then  $q(y) \in \mathcal{T}$ ,  $q(x) \in \mathcal{G}$ , and one of the following cases holds:*

- 1)  $|\text{supp}(y)| = |\text{supp}(x)| = 1$ ,  $x = ey$ .
- 2)  $|\text{supp}(y)| = |\text{supp}(x)| = 2$ ,  $y < x < ey$ .
- 3)  $|\text{supp}(y)| = 1$ ,  $|\text{supp}(x)| \geq 2$ ,  $y = x(J)$ .
- 4)  $|\text{supp}(y)| = 2$ ,  $|\text{supp}(x)| \geq 3$ ,  $y = x(J)$ .

*Proof.* a) We may assume that  $\text{supp}(x) = \{1, \dots, n\}$ . We have  $q(y) < q(x)$  because  $x$  is  $q$ -minimal. This forces  $q(y) \in \mathcal{T}$ ,  $q(x) \in \mathcal{G}$ .

b) Assume  $n = 1$ . Now  $y = y_1 \varepsilon_1$ ,  $x = x_1 \varepsilon_1$ , and  $\alpha_1^2 y_1 \in \mathcal{T}$ ,  $e \alpha_1^2 y_1 = \alpha_1^2 y_1 \in \mathcal{T}$ ,  $e \alpha_1^2 y_1 = \alpha_1^2 x_1$ . This implies  $x_1 = e y_1$ , whence  $x = e y$ .

c) Suppose that  $|J| = 1$ ,  $n \geq 2$ . We may assume that  $J = \{1\}$ . Now  $y = y_1\varepsilon_1$ ,  $\alpha_1 y_1^2 \in \mathcal{T}$  and  $y_1 \leq x_1$ , whence  $\alpha_1 y_1^2 \leq \alpha_1 x_1^2$ . Since  $q(y) \cong_\nu q(x)$ , the terms  $\alpha_1 x_1^2$  is  $\nu$ -dominant in the sum

$$\sum_1^n \alpha_i x_i^2 + \sum_{i < j} \beta_{ij} x_i x_j = q(x) \quad (7.16)$$

Since  $x$  is  $q$ -minimal, this forces  $\alpha_1 x_1^2 \in \mathcal{T}$  and then  $\alpha_1 y_1^2 = \alpha_1 x_1^2$ . We conclude that  $y_1 = x_1$ , i.e.,  $y = x(1)$ .

d) Suppose that  $|J| = 2$ ,  $n \geq 2$ . We may assume that  $J = \{1, 2\}$ . By Corollary 6.9,

$$\alpha_1 y_1^2 + \alpha_2 y_2^2 < \beta_{12} y_1 y_2 = q(y) \in \mathcal{T}.$$

It follows from  $q(y) \cong_\nu q(x)$  and  $y_1 \leq x_1$ ,  $y_2 \leq x_2$  that  $\beta_{12} x_1 x_2$  is a  $\nu$ -dominant term in the sum (7.16) and  $\beta_{12} x_1 x_2 \cong_\nu \beta_{12} y_1 y_2$ ,  $\beta_{12} x_1 x_2 \geq \beta_{12} y_1 y_2$ .

If  $n > 2$ , then the  $q$ -minimality of  $x$  forces  $\beta_{12} x_1 x_2 \in \mathcal{T}$ , and we conclude from  $y_1 \leq x_1$ ,  $y_2 \leq x_2$  that  $y_1 = x_1$ ,  $y_2 = x_2$ , i.e.,  $y = x(1, 2)$ .

If  $n = 2$ , we conclude from  $q(y) < q(x)$  that  $e\beta_{12} y_1 y_2 = \beta_{12} x_1 x_2$ , and then that  $y_1 \cong_\nu x_1$ ,  $y_2 \cong_\nu x_2$ , whence  $ex = ey$ . But  $x \neq ey$ , since the vector  $ey$  is not  $q$ -minimal. Thus either  $x_1 = ey_1$ ,  $x_2 = y_2$ , or  $x_1 = y_1$ ,  $x_2 = ey_2$ . We conclude that  $y < x < ey$ .  $\square$

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